ELECTROMAGNETIC SHOWERS

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Lecture 2

$$\frac{1}{\lambda_{\text{rad}}} = 4\alpha r_0^2 \, \frac{Z^2 \, N_A}{A} \, \log\left[183 \, Z^{-1/3}\right]$$

$$+ \text{ Electron Contribution.}$$

RADIATION LENGTH Meaning:

Length where the energy of an electron is reduced to E/e

7/9 of the mean free path of photons

$$\frac{1}{\lambda_{\rm rad}} = 4\alpha r_0^2 \, \frac{Z^2 \, N_A}{A} \; \log\left[183 \, Z^{-1/3}\right]$$

$$+ \text{ Electron Contribution.}$$

$$\frac{1}{\lambda_{\rm rad}} = \frac{4\alpha^3}{m_e^2} \, \frac{Z^2 \, N_A}{A} \, (\hbar \, c)^2 \; \log\left[\langle R_{\rm atom}\rangle \; \frac{m_e \, c}{\hbar}\right]$$

$$+ \text{ Electron Contribution.}$$

$$\frac{1}{\lambda_{\rm rad}} = \frac{4\alpha^3}{m_e^2} \frac{Z^2 N_A}{A} (\hbar c)^2 \log\left[\langle R_{\rm atom}\rangle \frac{m_e c}{\hbar}\right]$$

$$+ \text{ Electron Contribution.}$$

$$\langle R_{\rm hydrogen}\rangle \simeq a_{\rm Bohr} = \frac{\hbar^2}{m_e e^2} = \frac{1}{m_e \alpha} \frac{\hbar}{c} = \frac{137}{m_e} \frac{\hbar}{c}$$

$$\langle R_{\rm atom}\rangle \simeq 1.333\, a_{\rm Bohr}\, Z^{-1/3}$$

(large Z atom
Thomas-Fermi atom)

$$\langle R_{\rm atom}\rangle \simeq \frac{183}{m_e} \frac{\hbar}{c} Z^{-1/3}$$

From Particle Data Book

27.4. Photon and electron interactions in matter

27.4.1. ***Radiation length*** : High-energy electrons predominantly lose energy in matter by bremsstrahlung, and high-energy photons by e^+e^- pair production. The characteristic amount of matter traversed for these related interactions is called the radiation length X_0, usually measured in g cm^{-2}. It is both (a) the mean distance over which a high-energy electron loses all but $1/e$ of its energy by bremsstrahlung, and (b) $\frac{7}{9}$ of the mean free path for pair production by a high-energy photon [35]. It is also the appropriate scale length for describing high-energy electromagnetic cascades. X_0 has been calculated and tabulated by Y.S. Tsai [36]:

$$\frac{1}{X_0} = 4\alpha r_e^2 \frac{N_A}{A}\left\{Z^2\left[L_{\rm rad} - f(Z)\right] + Z\,L'_{\rm rad}\right\} . \qquad (27.20)$$

For $A = 1$ g mol^{-1}, $4\alpha r_e^2 N_A/A = (716.408 \text{ g cm}^{-2})^{-1}$. $L_{\rm rad}$ and $L'_{\rm rad}$ are given in Table 27.2. The function $f(Z)$ is an infinite sum, but for elements up to uranium can be represented to 4-place accuracy by

$$\begin{aligned} f(Z) = a^2\big[&(1+a^2)^{-1} + 0.20206 \\ &-0.0369\,a^2 + 0.0083\,a^4 - 0.002\,a^6\big] , \end{aligned} \qquad (27.21)$$

where $a = \alpha Z$ [37].

$$\frac{1}{X_0} = 4\alpha r_e^2 \frac{N_A}{A}\left\{Z^2\left[L_{\rm rad} - f(Z)\right] + Z\,L'_{\rm rad}\right\}$$

Table 27.2: Tsai's $L_{\rm rad}$ and $L'_{\rm rad}$, for use in calculating the radiation length in an element using Eq. (27.20).

Element	Z	$L_{\rm rad}$	$L'_{\rm rad}$
H	1	5.31	6.144
He	2	4.79	5.621
Li	3	4.74	5.805
Be	4	4.71	5.924
Others	> 4	$\ln(184.15\,Z^{-1/3})$	$\ln(1194\,Z^{-2/3})$

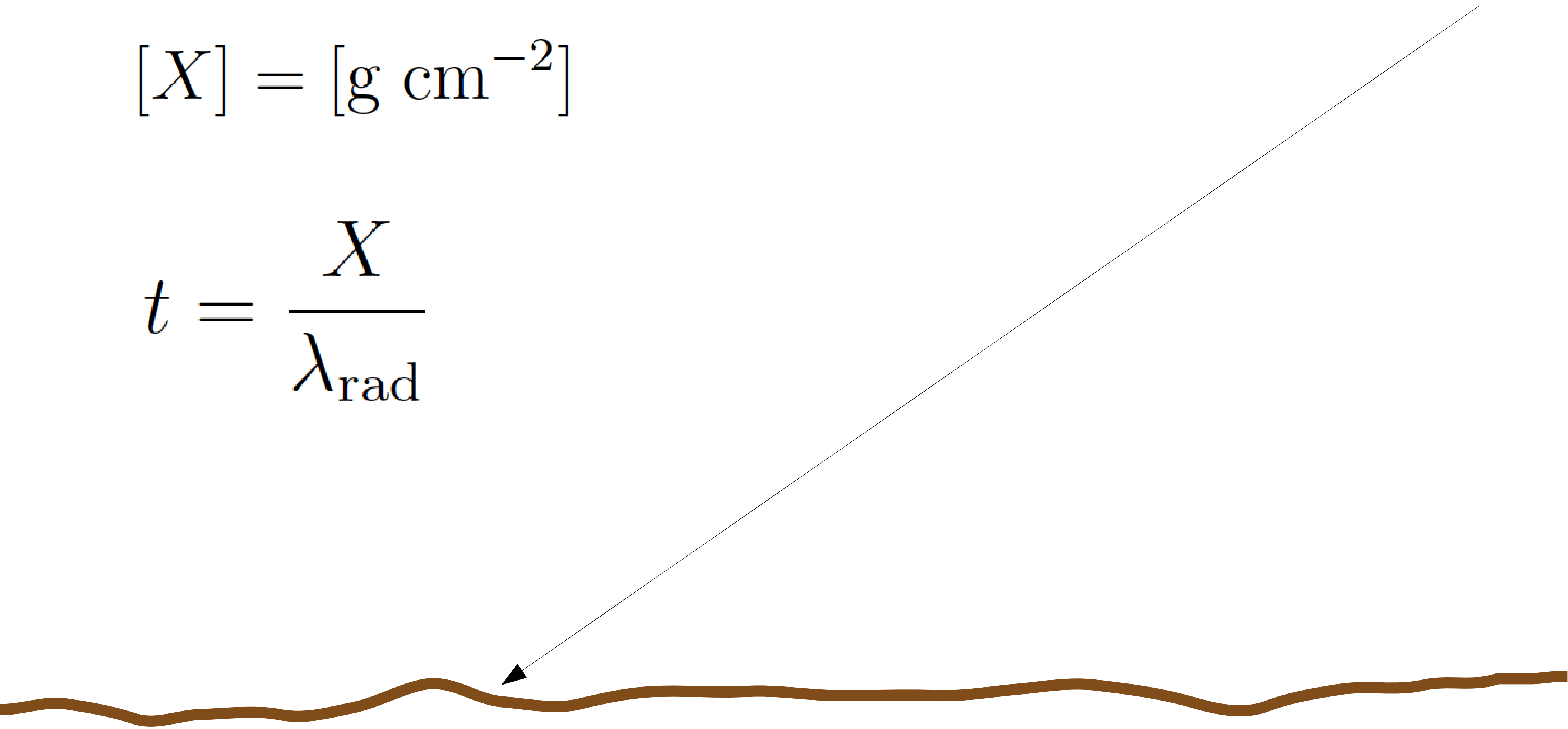
Longitudinal development of a shower.

$$X(z) = \int_0^z dz' \;\rho(z')$$

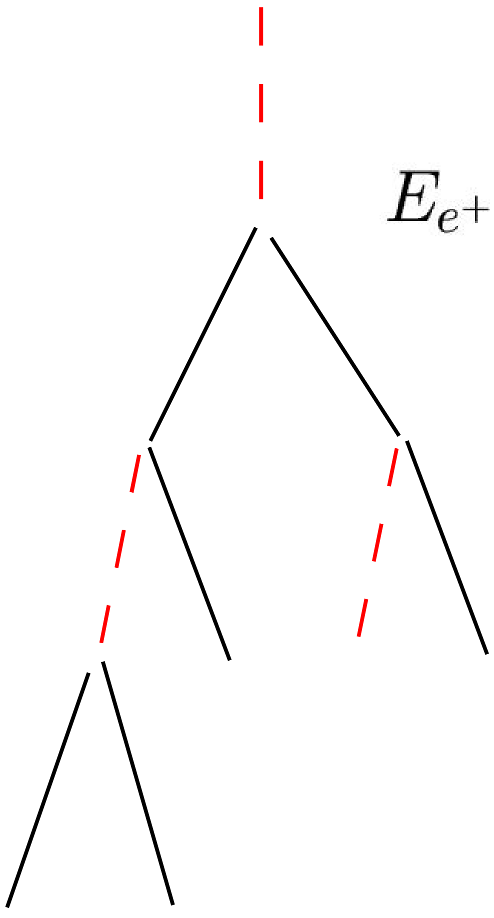
"Column density"

$$[X] = [\mathrm{g\ cm^{-2}}]$$

$$t = \frac{X}{\lambda_{\mathrm{rad}}}$$



ELECTROMAGNETIC SHOWERS



$$E_{e^+} = E_\gamma \, u$$

$$E_\gamma = E_e \, v$$

$\psi(u)$ Pair Production

$$e + Z \to e + \gamma + Z$$

$\varphi(v)$ Brems-strahlung

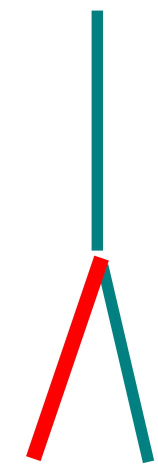
$$\gamma + Z \to e^+ + e^- + Z$$

Iteration of
2 fundamental processes
Pair Production
Bremsstrahlung

The "SPLITTING FUNCTIONS"

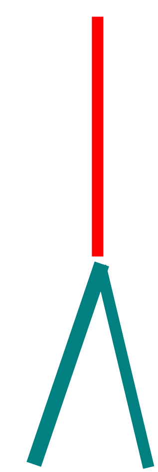
$$\varphi(v) = \left[\frac{d\sigma}{dv}(v)\right]_{\text{brems}} \left(\frac{N_A}{A}\,\lambda_{\text{rad}}\right)$$

$$\varphi(v) = \frac{1}{v}\,\left[1 - \left(\frac{2}{3} - 2\,b\right)\,(1-v) + (1-v)^2\right]$$



$$\psi(u) = \left[\frac{d\sigma}{du}(u)\right]_{\text{pair}} \left(\frac{N_A}{A}\,\lambda_{\text{rad}}\right)$$

$$\psi(u) = (1-u)^2 + \left(\frac{2}{3} - 2\,b\right)\,(1-u)\;u + u^2$$



$$\varphi(v) \quad dv\, dt$$

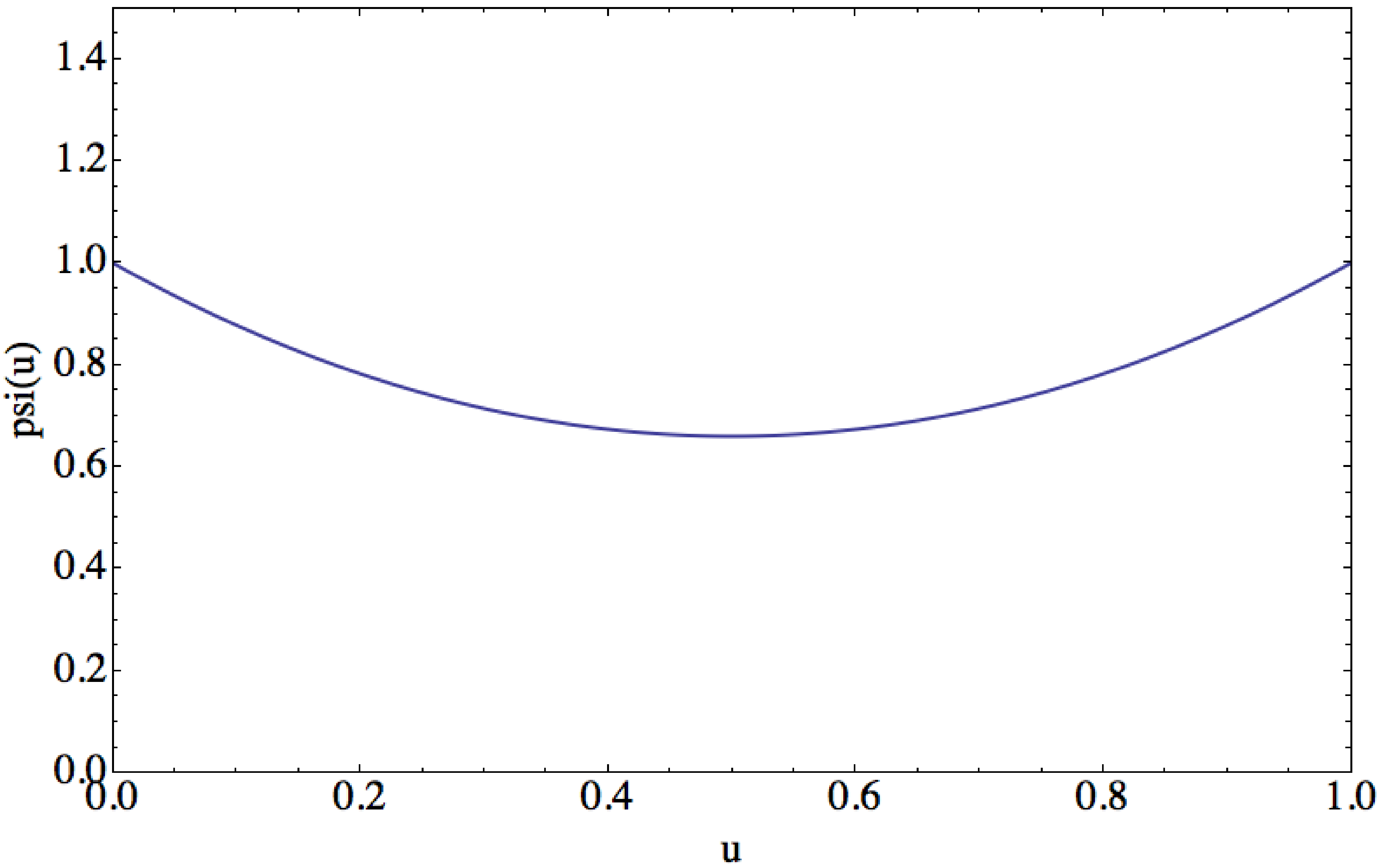
Probability for a photon of (any) energy E_γ
to generate one positron with fractional energy
$u = E_e/E_\gamma$ in the interval $[u, u + du]$
when traversing a layer of material of thickness dt

$$\psi(u) \quad dv\, dt$$

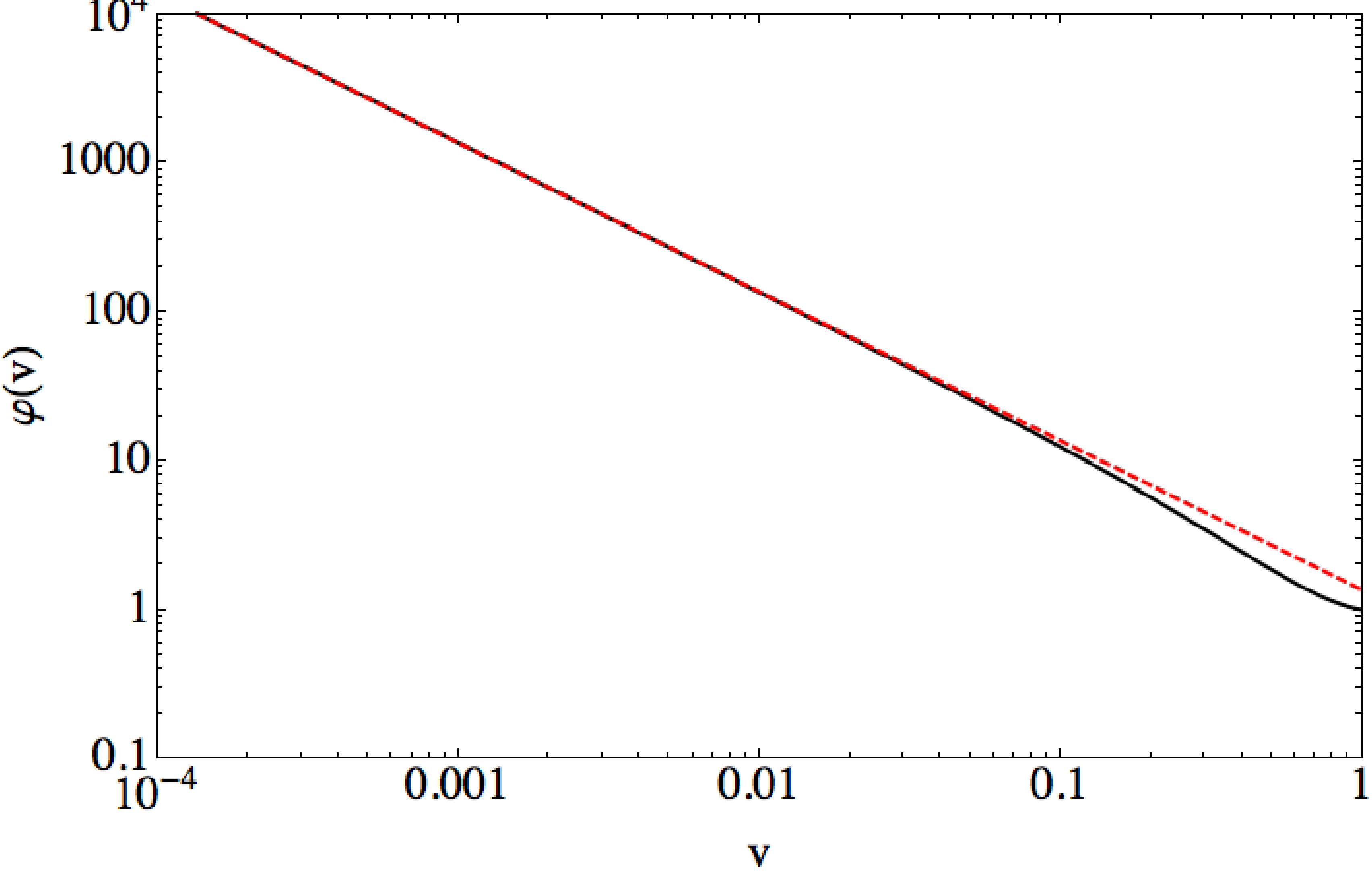
Probability for an electron of (any) energy E_e
to generate one photon with fractional energy
$v = E_\gamma/E_e$ in the interval $[v, v + dv]$,
when traversing a layer of material of thickness dt

$\psi(u)$

Pair Production

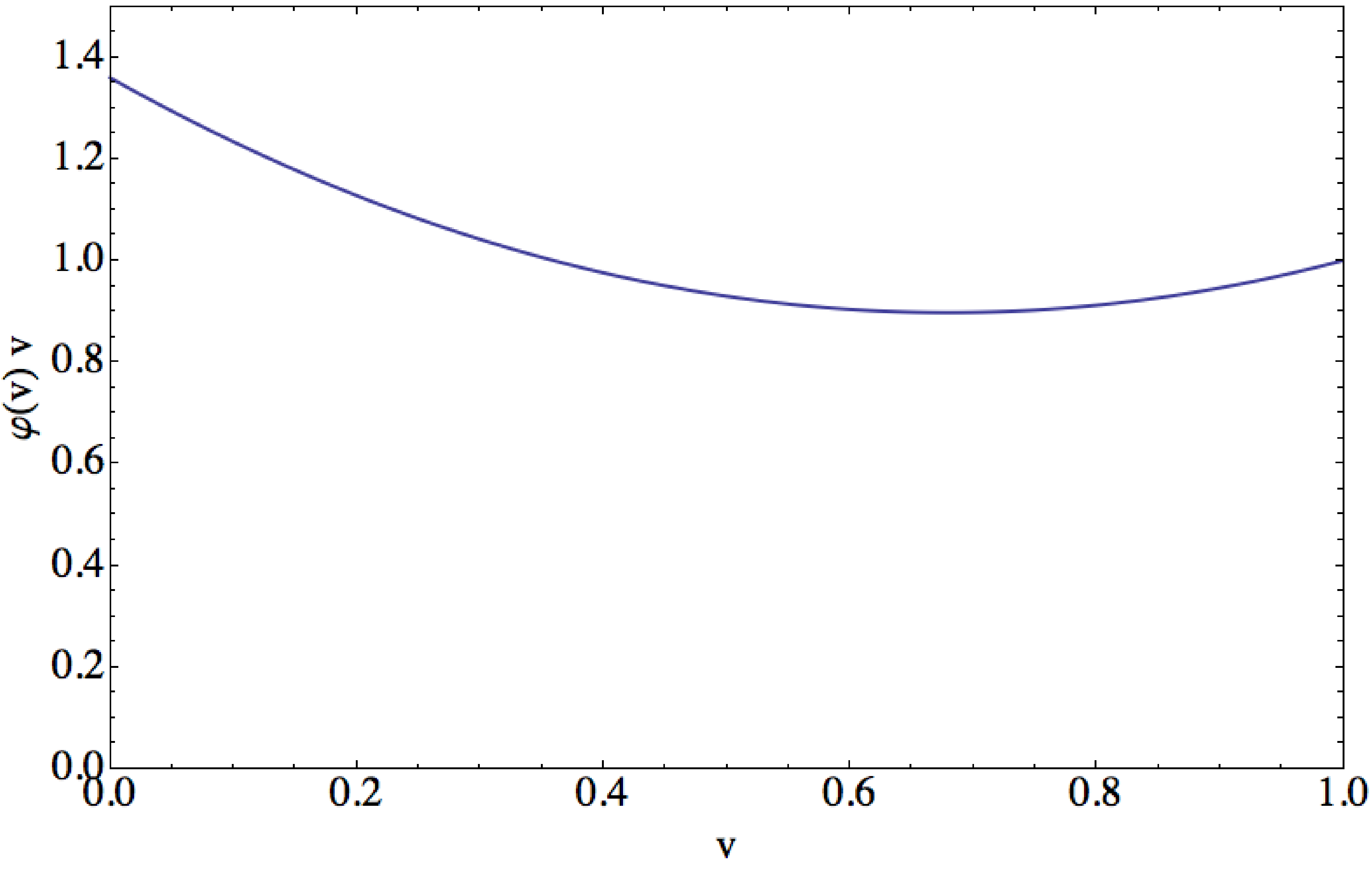


$\varphi(v)$ Bremsstrahlung



$\varphi(v)$

Bremsstrahlung

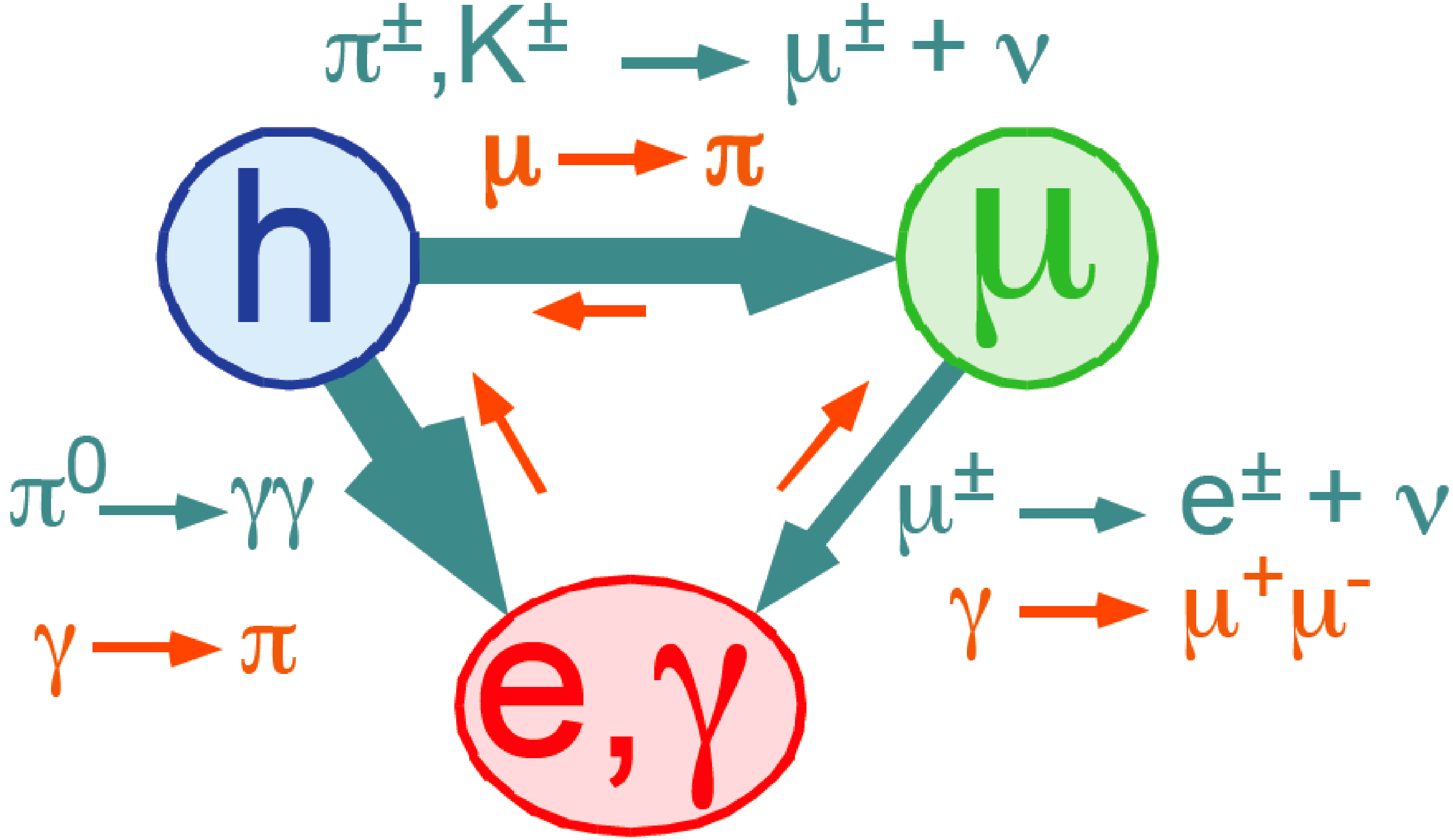


$$b \simeq \frac{1}{18\,\log(183\,Z^{-1/3})}$$

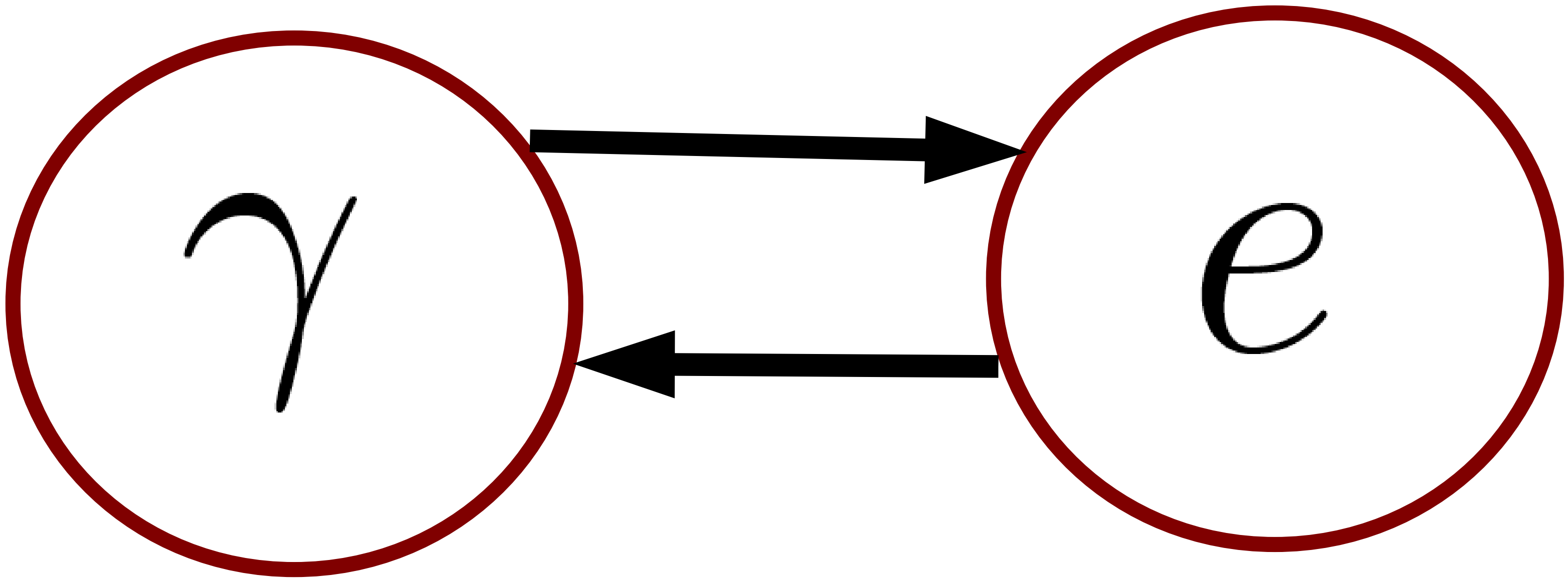
$$b \simeq 0.0135$$

$$\sigma_0 = \int_0^1 du\ \psi(u) = \frac{7}{9} - \frac{b}{3}$$

$$\int_0^1 dv\ \varphi(v) = 1 + b$$



π±,K± → μ± + ν
μ → π
h
μ
π0 → γγ
γ → π
μ± → e± + ν
γ → μ+μ-
e,γ


γ
e

Total ENERGY in a Shower

$$\mathcal{E}_{\rm shower}(t) = \mathcal{E}_{\rm electrons}(t) + \mathcal{E}_{\rm photons}(t)$$

$$\int_0^\infty dE\ E\ n_e(E,t) + \int_0^\infty dE\ E\ n_\gamma(E,t)$$

In Approximation A
the total Energy contained in Shower
is CONSTANT !

$$\lambda_{1,2}(s) = -\frac{1}{2}\left(A(s) + \sigma_0\right) \pm \frac{1}{2}\sqrt{\left(A(s) - \sigma_0\right)^2 + 4\,B(s)\,C(s)}$$

$$\begin{aligned}
A(s) &= \int_0^1 dv\;\varphi(v)\;\left[1 - (1-v)^s\right] \\
&= \left(\frac{4}{3} + 2\,b\right)\left(\frac{\Gamma'(1+s)}{\Gamma(1+s)} + \gamma\right) + \frac{s\;(7 + 5\,s + 12\,b\;(2+s))}{6\;(1+s)\;(2+s)} \\
B(s) &= 2\int_0^1 du\;u^s\;\psi(u) = \frac{2\;(14 + 11\,s + 3\,s^2 - 6\,b\;(1+s))}{3\;(1+s)\;(2+s)\;(3+s)} \\
C(s) &= \int_0^1 dv\;v^s\;\varphi(v) = \frac{8 + 7\,s + 3\,s^2 + 6\,b\;(2+s)}{3\,s\;(2 + 3\,s + s^2)}
\end{aligned}$$

AVERAGE LONGITUDINAL EVOLUTION for a PURELY ELECTRO-MAGNETIC SHOWER

$$n_e(E,t)$$

$$n_\gamma(E,t)$$

Two functions of energy and depth

Possible Generalizations:

3-Dimensional treatment.

$$n_{e,\gamma}(E, x, \theta_x, y, \theta_y, t)$$

Hadronic Showers: add other components

$$n_{p,n}(E, t) \qquad n_{\mu^\pm}(E, t)$$

$$n_{\pi^\pm}(E, t) \qquad n_{\nu}(E, t)$$

SYSTEM of
INTEGRO-DIFFERENTIAL EQUATIONS

that describe the evolution with **t** of

$n_e(E,t)$ $n_\gamma(E,t)$

for a given initial condition.

Variation with t of the number of photons with energy E

$$\frac{\partial n_\gamma}{\partial t}(E,t) = -n_\gamma(E,t) \int_0^1 du \; \psi(u)$$

$$+ \int_E^\infty dE' \int_0^1 dv \; n_e(E',t) \; \varphi(v) \; \delta[E - v\,E']$$

Variation with t of the number of photons with energy E

$$\frac{\partial n_\gamma}{\partial t}(E,t) = -n_\gamma(E,t) \int_0^1 du\, \psi(u)$$

$$+ \int_E^\infty dE' \int_0^1 dv\, n_e(E',t)\, \varphi(v)\, \delta[E - v\, E']$$

$$= -\sigma_0\, n_\gamma(E,t) \int_0^1 \frac{dv}{v}\, n_e\left(\frac{E}{v}, t\right)\, \varphi(v)$$

$(e \rightarrow \gamma)$

Electrons

$$\frac{\partial n_e}{\partial t}(E,t) = -n_e(E,t)\int_0^1 dv\ \varphi(v)$$

$$+\int_E^\infty dE' \int_0^1 dv\ n_e(E',t)\ \varphi(v)\ \delta[E-(1-v)\,E']$$

$$+\int_E^\infty dE' \int_0^1 du\ n_\gamma(E',t)\ \psi(u)\ \delta[E-u\,E']$$

Electrons

$$\frac{\partial n_e}{\partial t}(E,t) = -n_e(E,t) \int_0^1 dv\, \varphi(v)$$

$$+\int_E^\infty dE' \int_0^1 dv\, n_e(E',t)\, \varphi(v)\, \delta[E-(1-v)\,E']$$

$(\gamma \to e)$

$$+\int_E^\infty dE' \int_0^1 du\, n_\gamma(E',t)\, \psi(u)\, \delta[E-u\,E']$$

$$2 \int_0^1 \frac{du}{u}\, n_\gamma\left(\frac{E}{u},t\right)\, \psi(u)$$

Electrons

$(e \to e)$

$$\frac{\partial n_e}{\partial t}(E,t) = -n_e(E,t) \int_0^1 dv\, \varphi(v)$$

$$+ \int_E^\infty dE' \int_0^1 dv\, n_e(E',t)\, \varphi(v)\, \delta[E - (1-v)\, E']$$

$$\int_0^1 \frac{dv}{1-v}\, n_e\left(\frac{E}{1-v}, t\right)\, \varphi(v)$$

$$+ \int_E^\infty dE' \int_0^1 du\, n_\gamma(E',t)\, \psi(u)\, \delta[E - u\, E']$$

Electrons

$(e \to e)$

$$\frac{\partial n_e}{\partial t}(E,t) = -n_e(E,t) \int_0^1 dv \, \varphi(v)$$

$$+ \int_E^\infty dE' \int_0^1 dv \, n_e(E',t) \, \varphi(v) \, \delta[E - (1-v)\,E']$$

$$\int_0^1 \frac{dv}{1-v} \, n_e\left(\frac{E}{1-v}, t\right) \varphi(v)$$

$$+ \int_E^\infty dE' \int_0^1 du \, n_\gamma(E',t) \, \psi(u) \, \delta[E - u\,E']$$

2 divergent e -> e contributions.
Their combination is finite.

$$\frac{\partial n_e(E,t)}{\partial t} = -\int_0^1 dv\ \ \varphi(v)\ \left[n_e(E,t) - \frac{1}{1-v}\, n_e\left(\frac{E}{1-v}, t\right)\right]$$

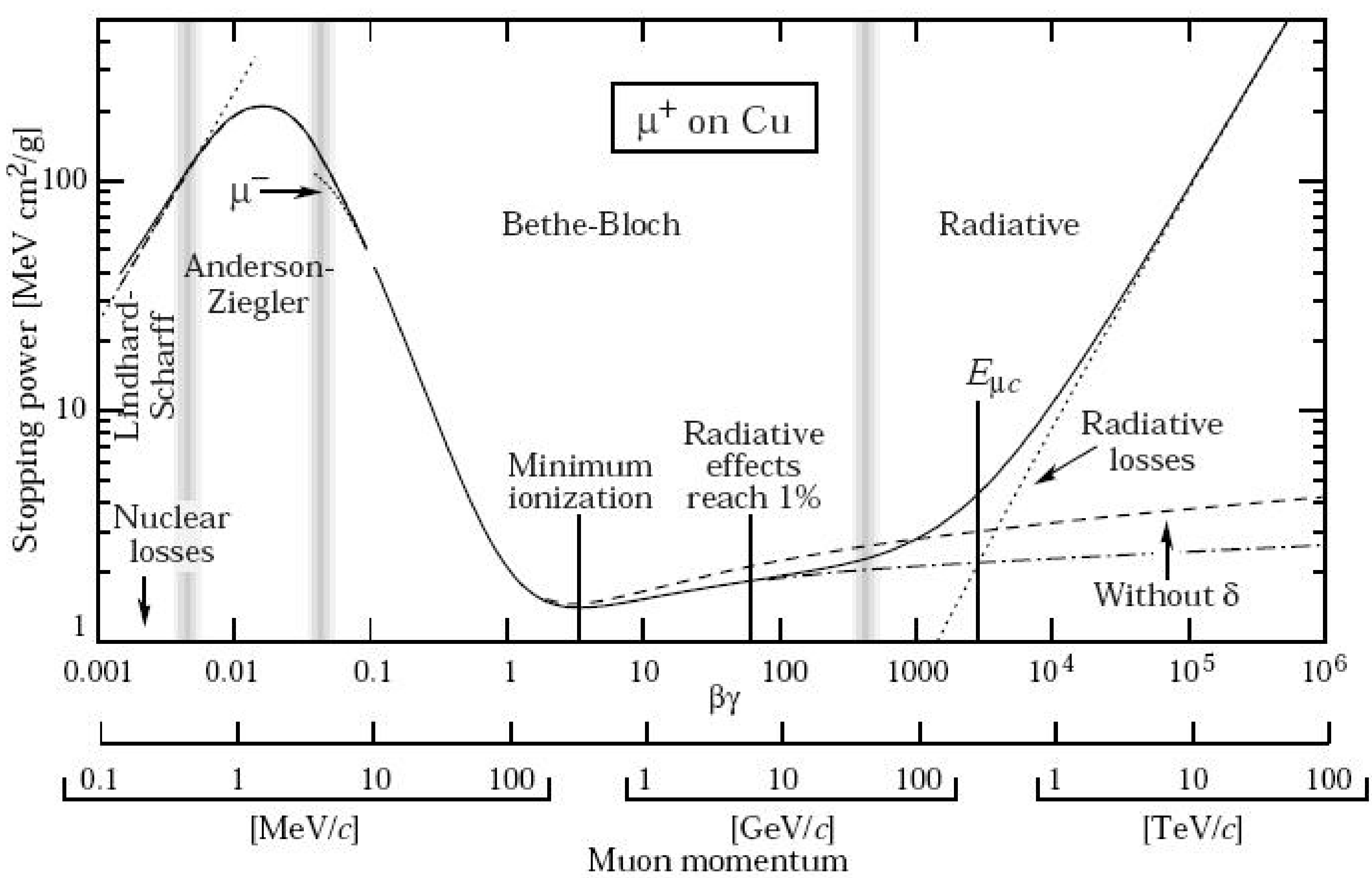
$$+2\int_0^1 \frac{du}{u}\, \psi(u)\, n_\gamma\left(\frac{E}{u}, t\right)$$

$$\frac{\partial n_\gamma(E,t)}{\partial t} = \int_0^1 \frac{dv}{v}\, \varphi(v)\, n_e\left(\frac{E}{v}, t\right) - \sigma_0\, n_\gamma(E,t)\ .$$

Approximation A

INTRODUCTION of
ENERGY LOSS of ELECTRONS for COLLISIONS

INTRODUCTION of
ENERGY LOSS of ELECTRONS for COLLISIONS



Bethe-Bloch formula

$$-\frac{dE}{dx} = Kz^2 \frac{Z}{A} \frac{1}{\beta^2} \left[\frac{1}{2} \ln \frac{2m_e c^2 \beta^2 \gamma^2 T_{\max}}{I^2} - \beta^2 - \frac{\delta(\beta\gamma)}{2} \right]$$

Very Simple Approximation:
ENERGY INDEPENDENT LOSS

$$\left.\frac{dE}{dt}\right|_{\text{collisions}} = -\varepsilon$$

Critical energy

Insert the collision energy loss in the shower equations:
General treatment:

$$\frac{dE}{dt} = \beta(t)$$

Energy variation Law

$n(E, t)$ $\qquad$ $n(E, t + dt)$

$$n(E, t + dt)\, dE = n(E', dt)\, dE'$$

$$E' = E - \beta(E)\, dt \qquad dE' = \left(1 - \frac{d\beta(E)}{dE}\right)\, dE$$

$$n(E, t + dt)\ dE = n(E', dt)\ dE'$$

$$\left[n(E,t) + \frac{\partial n(E,t)}{\partial t}\,dt\right]\ dE \qquad =$$

$$\left[n(E,t) + \frac{\partial n(E,t)}{\partial E}\,\beta(E)\,dt\right]\left(1 - \frac{d\beta(E)}{dE}\right)\ dE$$

$$n(E, t + dt)\; dE = n(E', dt)\; dE'$$

$$\left[n(E,t) + \frac{\partial n(E,t)}{\partial t}\, dt \right] \; dE \qquad =$$

$$\left[n(E,t) + \frac{\partial n(E,t)}{\partial E}\, \beta(E)\, dt \right] \left(1 - \frac{d\beta(E)}{dE} \right) \; dE$$

$$\frac{\partial n(E,t)}{\partial t} = -\frac{\partial}{\partial E} \; [n(E,t)\; \beta(E)]$$

$$\frac{\partial n_e(E,t)}{\partial t} = -\int_0^1 dv\ \varphi(v)\left[n_e(E,t) - \frac{1}{1-v}\, n_e\left(\frac{E}{1-v}, t\right)\right]$$

$$+2\int_0^1 \frac{du}{u}\,\psi(u)\, n_\gamma\left(\frac{E}{u}, t\right)$$

$$\frac{\partial n_\gamma(E,t)}{\partial t} = \int_0^1 \frac{dv}{v}\,\varphi(v)\, n_e\left(\frac{E}{v}, t\right) - \sigma_0\, n_\gamma(E,t)$$

Approximation A

$$\frac{\partial n_e(E,t)}{\partial t} = -\int_0^1 dv \;\; \varphi(v) \left[n_e(E,t) - \frac{1}{1-v}\, n_e\left(\frac{E}{1-v}, t\right)\right]$$

$$+2 \int_0^1 \frac{du}{u}\, \psi(u)\, n_\gamma\left(\frac{E}{u}, t\right)$$

$$+\,\varepsilon\, \frac{\partial n_e(E,t)}{\partial E}$$

$$\frac{\partial n_\gamma(E,t)}{\partial t} = \int_0^1 \frac{dv}{v}\, \varphi(v)\, n_e\left(\frac{E}{v}, t\right) - \sigma_0\, n_\gamma(E,t)$$

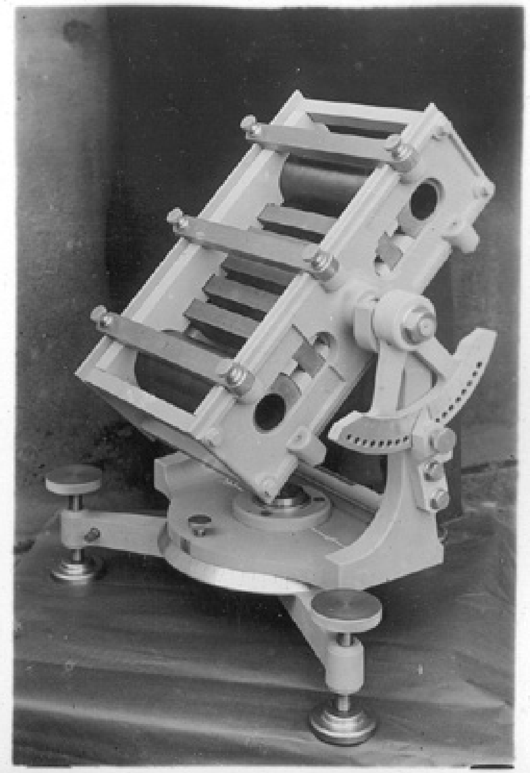
Approximation B

OCTOBER, 1941 REVIEWS OF MODERN PHYSICS

Cosmic-Ray Theory

BRUNO ROSSI AND KENNETH GREISEN
Cornell University, Ithaca, New York

Bruno Rossi
1933 Eritrea
East West effect



Kenneth Greisen
NCAR Texas 1971

after discovery of
200 MeV photons from
the Crab Nebula

Shower Equations in "Approximation A"

(neglect electron ionization losses)

$$n_e(E,t)$$
$$n_\gamma(E,t)$$

$$\frac{\partial n_e(E,t)}{\partial t} = -\int_0^1 dv \;\; \varphi_0(v) \left[n_e(E,t) - \frac{1}{1-v} \;\; n_e\left(\frac{E}{1-v}, t\right)\right]$$

$$+ 2 \int_0^1 \frac{du}{u} \, \psi(u) \, n_\gamma \left(\frac{E}{u}, t\right)$$

No Parameters with the Dimension of Energy

$$\frac{\partial n_\gamma(E,t)}{\partial t} = \int_0^1 \frac{dv}{v} \, \varphi(v) \, n_e\left(\frac{E}{v}, t\right) - \sigma_0 \, n_\gamma(E,t)$$

Solutions to the shower equations.

Initial Condition:

$$\begin{cases} n_e(E,0) & = & 0 \\ n_\gamma(E,0) & = & \delta[E - E_0] \end{cases}$$

Photon of energy E_0

$$\begin{cases} n_e(E,0) & = & \delta[E - E_0] \\ n_\gamma(E,0) & = & 0 \end{cases}$$

Electron of energy E_0

Let us consider an electron population
that is has the spectral shape of
an unbroken power law and no photons:

$$\begin{cases} n_e(E,0) & = & K\ E^{-(s+1)} \\ n_\gamma(E,0) & = & 0 \end{cases}$$

Study the Shower evolution
using approximation A

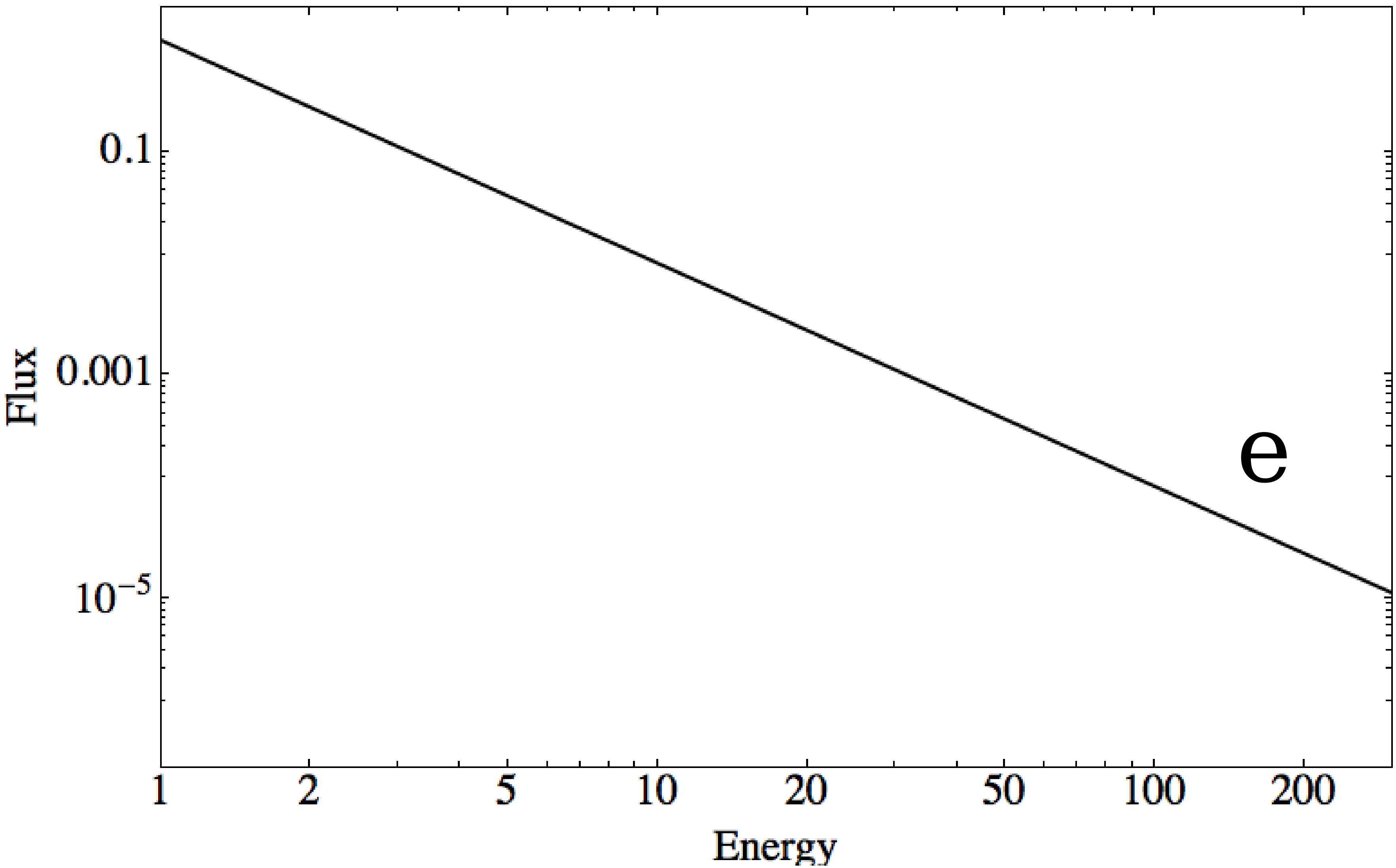
$$\begin{cases} n_e(E,0) & = & K\ E^{-(s+1)} \\ n_\gamma(E,0) & = & 0 \end{cases}$$

Initial condition

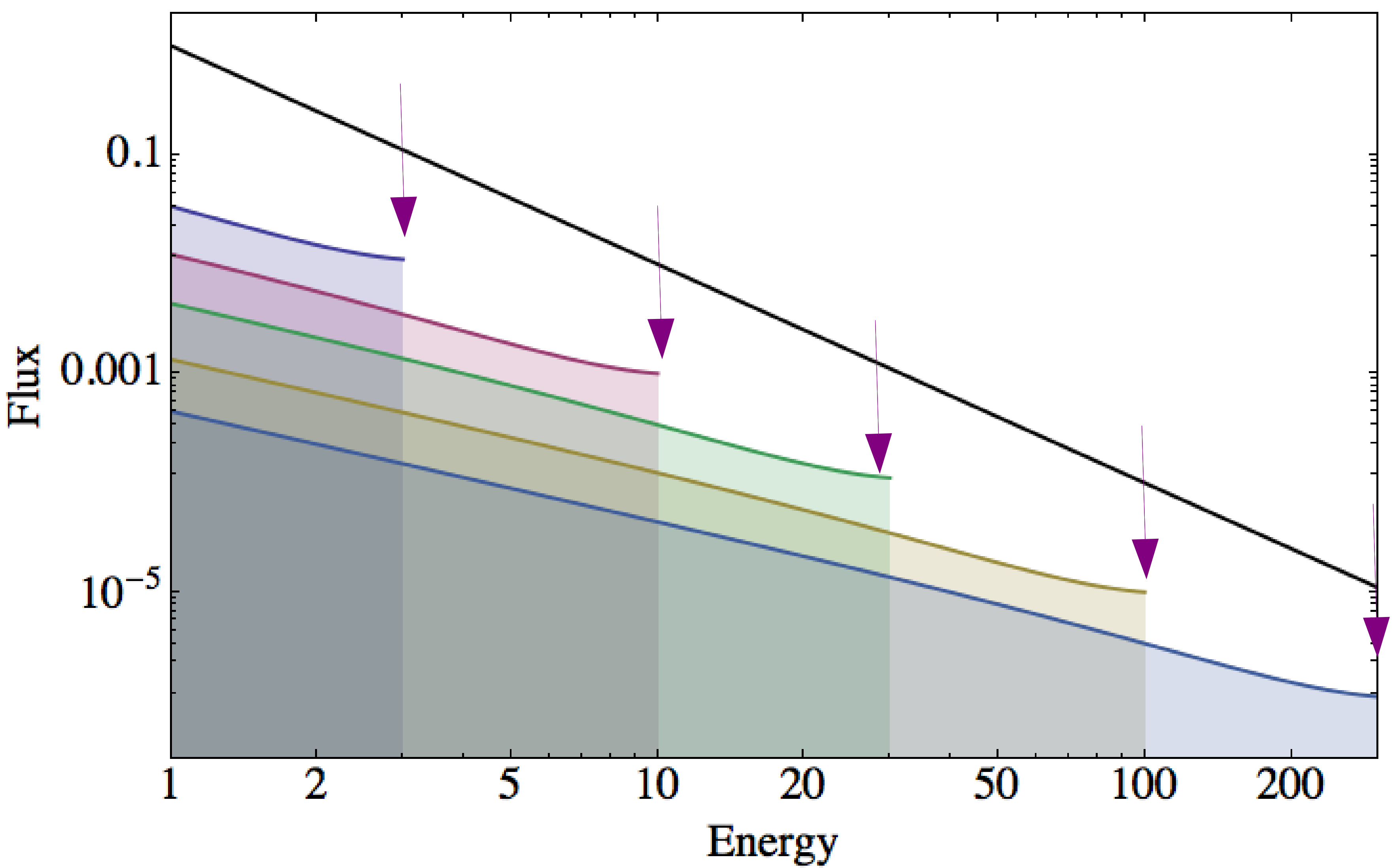
Electron and Photon population
remain a power law of same slope
Only the normalizations are a function of the depth t

$$\begin{cases} n_e(E,t) & = & K_e(t)\ E^{-(s+1)} \\ n_\gamma(E,t) & = & K_\gamma(t)\ E^{-(s+1)} \end{cases}$$

Depth evolution



0.1
0.001
10^{-5}
Flux
1
2
5
10
20
50
100
200
Energy
e



0.1
0.001
10^{-5}
Flux
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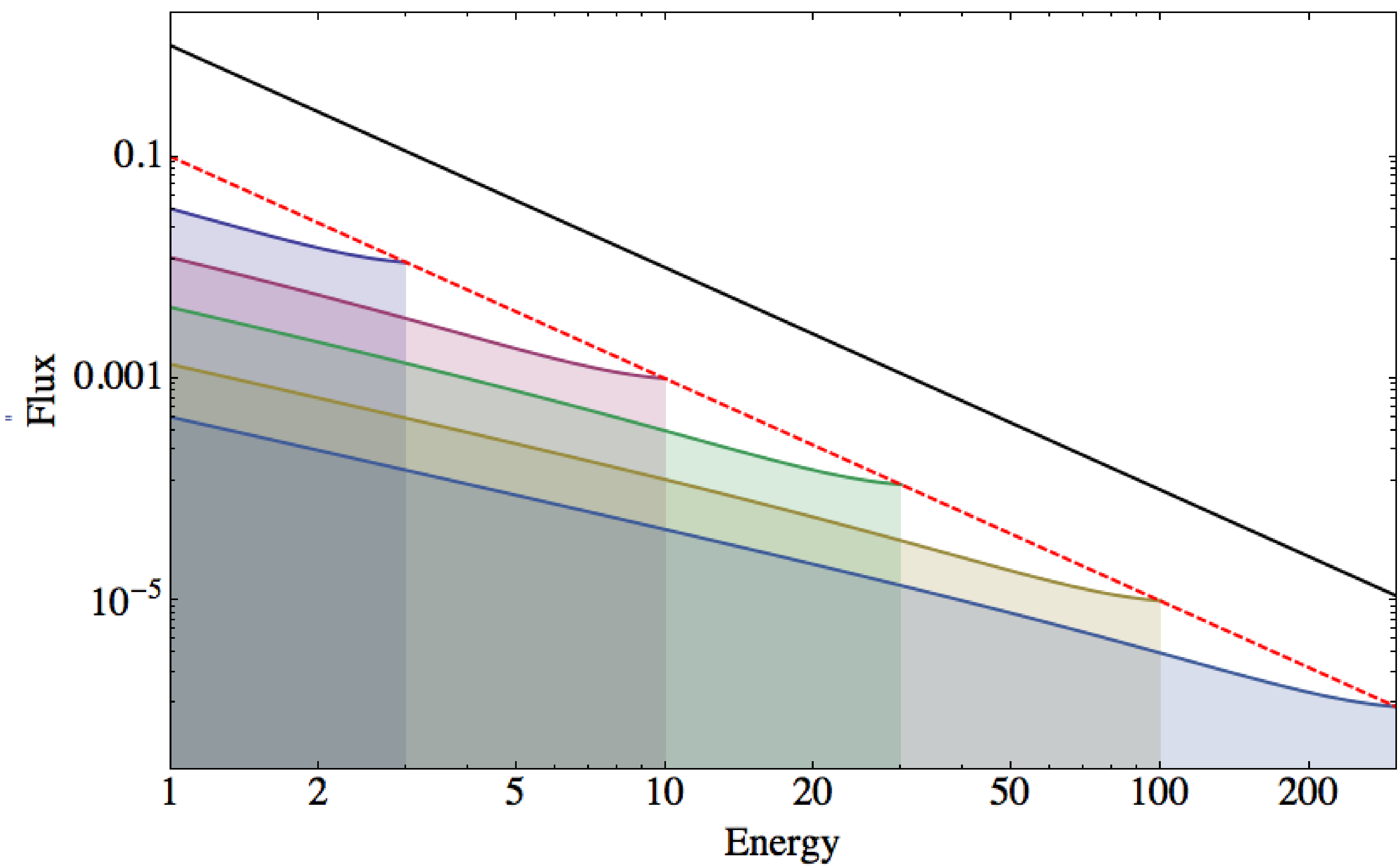

0.1
0.001
10^{-5}
Flux
1
2
5
10
20
50
100
200
Energy

$$\begin{cases} n_e(E,t) = K_e(t)\ E^{-(s+1)} \\ n_\gamma(E,t) = K_\gamma(t)\ E^{-(s+1)} \end{cases}$$

Coefficients $K_{e,\gamma}(t)$ are linear combinations of two exponential

$$K_{e,\gamma}(t) = a_{e,\gamma}\ e^{\lambda_1(s)\,t} + b_{e,\gamma}\ e^{\lambda_2(s)\,t}$$

$$\begin{cases} n_e(E,t) = K_e(t)\ E^{-(s+1)} \\ n_\gamma(E,t) = K_\gamma(t)\ E^{-(s+1)} \end{cases}$$

$$K_{e,\gamma}(t) = a_{e,\gamma}\ e^{\lambda_1(s)\,t} + b_{e,\gamma}\ e^{\lambda_2(s)\,t}$$

One controls the (faster) convergence to an s-dependent gamma/e ratio (large and negative)

$\lambda_2(s)$

A second exponential describes the (slower) evolution of the two population with a constant ratio.

$\lambda_1(s)$

$$\begin{cases} n_e(E,t) = K_e\ E^{-2} \\ n_\gamma(E,t) = K_\gamma\ E^{-2} \end{cases}$$

S = 1

Special spectrum

Equal amount of energy per decade of E

$$\begin{cases} n_e(E,t) = K_e\ E^{-2} \\ n_\gamma(E,t) = K_\gamma\ E^{-2} \end{cases}$$

S = 1

Special spectrum

Equal amount of energy per decade of E

$$t \to t + dt$$

$$dn_e = -dn_\gamma = \left(-n_e\ \langle v \rangle + n_\gamma\ \sigma_0\right)\ dt$$

$$\frac{n_\gamma}{n_e} = \frac{\langle v \rangle}{\sigma_0} \iff dn_e = dn_\gamma = 0$$

depth-independent solution

What can we say about: $\lambda_1(s)$

Without explicit calculation ?

$$s = 1 \iff \lambda_1(s) = 0$$

Spectrum E^{-2} equal power per decade of E

Pair Production and Bremsstrahlung
"redistribute the energy"
but "nothing can change"

What can we say about: $\lambda_1(s)$

$$s = 1 \iff \lambda_1(s) = 0$$

$$s < 1 \iff \lambda_1(s) > 0$$

Spectrum flatter than E^{-2}
power per decade of E grows with E

What can we say about: $\lambda_1(s)$

$$s = 1 \iff \lambda_1(s) = 0$$

$$s < 1 \iff \lambda_1(s) > 0$$

$$s > 1 \iff \lambda_1(s) < 0$$

Spectrum steeper than E^{-2}
power per decade of E decreases with E

Insert functional form of the solution in the shower equation.

$$\begin{cases} n_e(E,t) = K_e \; E^{-(s+1)} \; e^{\lambda t} \\ n_\gamma(E,t) = K_\gamma \; E^{-(s+1)} \; e^{\lambda t} \end{cases}$$

$$\frac{\partial n_e(E,t)}{\partial t} = -\int_0^1 dv \;\; \varphi(v) \left[n_e(E,t) - \frac{1}{1-v} n_e \left(\frac{E}{1-v}, t \right) \right]$$

$$+2 \int_0^1 \frac{du}{u} \psi(u) \, n_\gamma \left(\frac{E}{u}, t \right)$$

$$\frac{\partial n_\gamma(E,t)}{\partial t} = \int_0^1 \frac{dv}{v} \varphi(v) \, n_e \left(\frac{E}{v}, t \right) - \sigma_0 \, n_\gamma(E,t) \; .$$

Obtain simple quadratic equation connecting

s λ K_e/K_γ

$$\frac{\partial n_e(E,t)}{\partial t} \to \lambda\, n_e(E,t)$$

Time derivative

$$2\int_0^1 \frac{du}{u}\,\psi(u)\,n_\gamma\left(\frac{E}{u},t\right)$$

Example of one term

$$2\int_0^1 \frac{du}{u}\,\psi(u)\left[K_\gamma\left(\frac{E}{u}\right)^{-(s+1)} e^{\lambda t}\right]$$

$$2\,K_\gamma\,E^{-(s+1)}\,e^{\lambda t)}\int_0^1 \frac{du}{u}\,\psi(u)\,u^{(s+1)}$$

$$2K_\gamma\,E^{-(s+1)}\,e^{\lambda t}\;B(s)$$