Distribution of Bipartite Entanglement of a Random Pure State

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J. Stat. Phys. 142, 403 (2011) (long version)

Plan

Plan:

- Bipartite entanglement of a random pure state

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- Reduced density matrix $\longrightarrow$ random matrix theory

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  Coulomb gas technique for large systems

  Phase transitions

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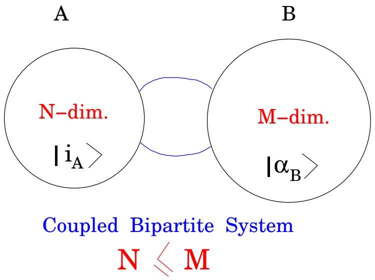
- Summary and Conclusions

Coupled Bipartite System



Bipartite quantum system $A \times B$: Hilbert space $H_A \otimes H_B$

subsystem A: dimension N (the system to study)

subsystem B: dimension $M \geq N$ ("environment")

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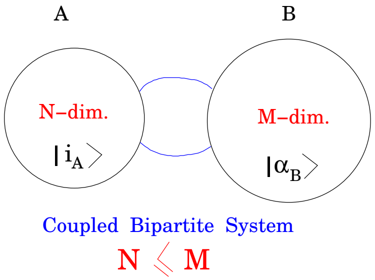

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subsystem A: dimension N (the system to study)

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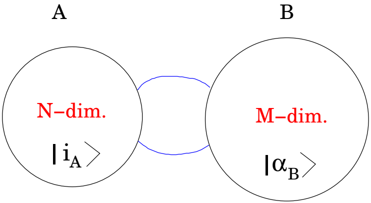
Large system: limit $N \gg 1$ and $M \gg 1$ with $M \approx N$.

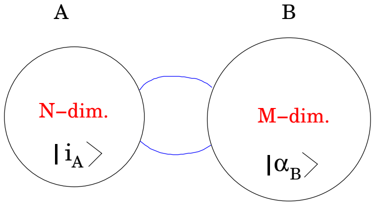
Pure State and Reduced Density Matrix



Coupled Bipartite System

$N \leqslant M$

Pure State and Reduced Density Matrix



Coupled Bipartite System

$$\mathbf{N} \leqslant \mathbf{M}$$

Any pure state of the full system: $$|\psi\rangle = \sum_{i,\alpha} x_{i,\alpha}\,|i_A\rangle \otimes |\alpha_B\rangle$$

$X = [x_{i,\alpha}] \rightarrow (N \times M)$ rectangular Coupling matrix

Pure State of Bipartite System

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- $$|\psi>=\sum_{i,\alpha} x_{i,\alpha}\,|i_A>\otimes|\alpha_B>$$

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$$|\psi>=\sum_i a_i\,|i_A>\otimes\sum_\alpha b_\alpha\,|\alpha_B>=|\Phi_A>\otimes|\Phi_B>$$

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$$\hat{\rho} = |\psi >< \psi| \quad \text{with } \mathrm{Tr}[\hat{\rho}] = 1$$

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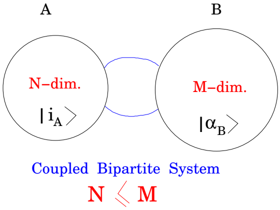
- Pure state: $\hat{\rho} \neq \sum_k p_k |\psi_k >< \psi_k| \rightarrow$ not a Mixed state

Reduced Density Matrix of subsystem A:



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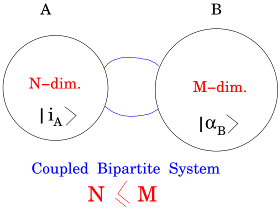

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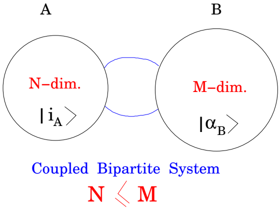

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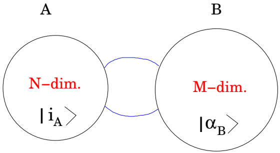

Coupled Bipartite System

$N \lll M$

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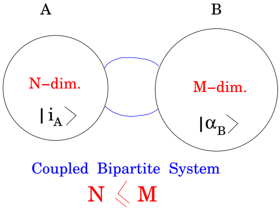

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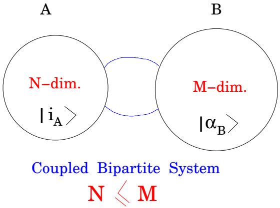

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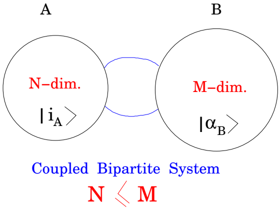

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  $$= \sum_{i,j=1}^{N} W_{i,j}\, |i_A >< j_A|$$

  where the $N \times N$ matrix: $\boxed{W = XX^{\dagger}}$

Eigenvalues of W:

- In the diagonal representation

$$\hat{\rho}_A = \sum_{i,j=1}^{N} W_{i,j} \, |i_A >< j_A| \to \sum_{i=1}^{N} \lambda_i |\lambda_i^A >< \lambda_i^A|$$

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Entanglement

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$0\le\lambda_i\le 1$ and $\sum_{i=1}^{N}\lambda_i=1$

(i) **Unentangled**

$\lambda_{i_0}=1,\;\lambda_j=0\;\forall j\neq i_0$

$|\psi\rangle=|\lambda_{i_0}^A\rangle\otimes|\lambda_{i_0}^B\rangle$

is **separable**

$\hat{\rho}_A=|\lambda_{i_0}^A\rangle\langle\lambda_{i_0}^A|$ is pure

(ii) **Maximally entangled**

$\lambda_j=1/N$ for all j

(all eigenvalues equal)

$|\psi\rangle$ is a superposition of all product states

$\hat{\rho}_A=\frac{1}{N}\sum_{i=1}^{N}|\lambda_i^A\rangle\langle\lambda_i^A|=\frac{1}{N}\mathbf{1}_A$ is completely mixed

Entanglement entropy

Subsystem A described by $\hat{\rho}_A = \sum_{i=1}^{N} \lambda_i |\lambda_i^A\rangle\langle\lambda_i^A|$

- **Von Neuman entropy**: $S_{\mathrm{VN}} = -\mathrm{Tr}\left[\hat{\rho}_A \ln \hat{\rho}_A\right] = -\sum_{i=1}^{N} \lambda_i \ln \lambda_i$

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- $S_{q\to 1} = S_{\rm VN}$ and $S_{q\to\infty} = -\ln(\lambda_{\max})$
- $\Sigma_2 = \exp[-S_{q=2}] \longrightarrow$ **Purity**

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(i) **Unentangled**
$\lambda_{i_0} = 1,\ \lambda_j = 0\ \forall j \neq i_0$

$\hat{\rho}_A = |\lambda_{i_0}^A\rangle\langle\lambda_{i_0}^A|$ is pure

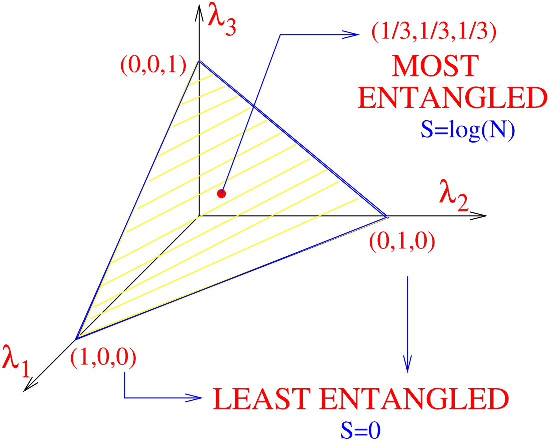
$S_{\mathrm{VN}} = 0 = S_q$ is **minimal**

(ii) **Maximally entangled**
$\lambda_j = 1/N$ for all j

$\hat{\rho}_A = \frac{1}{N}\sum_{i=1}^{N} |\lambda_i^A\rangle\langle\lambda_i^A|$ mixed

$S_{\mathrm{VN}} = \ln N = S_q$ is **maximal**

A Simple Diagram for N=3



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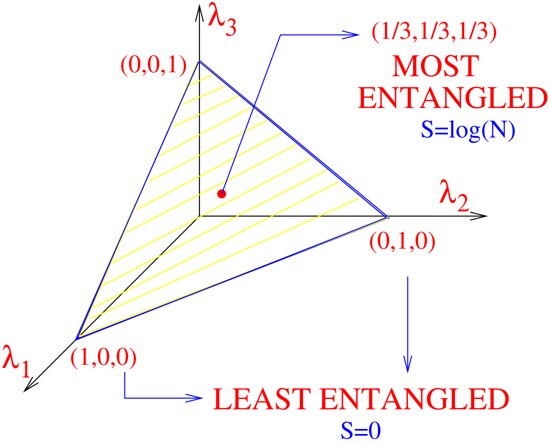

Random Pure State: Haar Measure

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• random pure state $\longrightarrow$ where $X = [x_{i,\alpha}]$ are **uniformly distributed** among the sets of $\{x_{i,\alpha}\}$ satisfying $\sum_{i,\alpha} |x_{i,\alpha}|^2 = \mathrm{Tr}\rho = 1$

$\longrightarrow$ Haar measure

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- Equivalently, for the $N \times M$ matrix

$$P(X) \propto \delta\left(\mathrm{Tr}(XX^\dagger) - 1\right)$$

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- In the basis $|i^A\rangle$ of H_A: $\hat{\rho}_A = W = XX^\dagger$

$\longrightarrow$ Distribution of the eigenvalues λ_i of $\hat{\rho}_A$?

$\longrightarrow$ **Distribution of the entanglement entropies** S_{VN}, S_q?

Joint PDF of Eigenvalues

- the joint pdf $P(\lambda_1, \lambda_2, \ldots, \lambda_N)$
  where $\{\lambda_1, \lambda_2, \ldots, \lambda_N\} \to$ eigenvalues of the Wishart matrix

  $W = XX^\dagger$ ($X \to N \times M$ Gaussian random matrix)

  with an additional constraint $\boxed{\sum_{i=1}^{N} \lambda_i = 1}$

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- Joint distribution of Wishart eigenvalues (James '64):

$$\boxed{P(\{\lambda_i\}) \propto \exp\left[-\frac{\beta}{2}\sum_{i=1}^{N}\lambda_i\right] \prod_i \lambda_i^{\frac{\beta}{2}(1+M-N)-1} \prod_{j<k} |\lambda_j - \lambda_k|^\beta}$$

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(Llyod & Pagels '88, Zyczkowski & Sommers '2001)

Joint PDF of Eigenvalues

- the joint pdf $P(\lambda_1, \lambda_2, \ldots, \lambda_N)$
  where $\{\lambda_1, \lambda_2, \ldots, \lambda_N\} \to$ eigenvalues of the Wishart matrix

  $W = XX^\dagger$ ($X \to N \times M$ Gaussian random matrix)

  with an additional constraint $\boxed{\sum_{i=1}^{N} \lambda_i = 1}$

- Joint distribution of Wishart eigenvalues (James '64):

$$\boxed{P(\{\lambda_i\}) \propto \exp\left[-\frac{\beta}{2}\sum_{i=1}^{N}\lambda_i\right]\prod_i \lambda_i^{\frac{\beta}{2}(1+M-N)-1}\prod_{j<k}|\lambda_j-\lambda_k|^\beta} \times \delta\left(\sum_{i=1}^{N}\lambda_i - 1\right)$$

(Llyod & Pagels '88, Zyczkowski & Sommers '2001)

- Given this pdf, what is the distribution of the Renyi entropy:

$$S_q = \frac{1}{1-q}\ln\left[\sum_{i=1}^{N}\lambda_i^q\right]$$

Random Pure State: Well Studied

- Average von Neumann entropy: $\langle S_{\mathrm{VN}} \rangle \to \ln N - \frac{N}{2M}$ for $M \sim N \gg 1$

$\longrightarrow$ **Near Maximal**

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- Laplace transform of the purity ($q = 2$) distribution for large N [Facchi et. al. (2008, 2010)]

Results: Entropy Distribution

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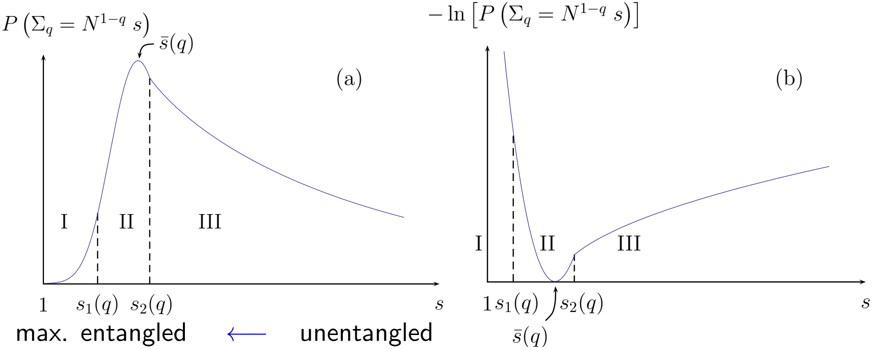
For $q = 2$, $\langle \Sigma_2 \rangle \approx \frac{2}{N}$ (purity); For $q \to 1$, $\langle S_{VN} \rangle \approx \ln N - \frac{1}{2}$

Results: pdf of $\Sigma_q = \sum_i \lambda_i^q$

$$P\left(\Sigma_q = N^{1-q}\, \mathbf{s}\right) \approx \begin{cases} \exp\left\{-\beta N^2 \Phi_I(s)\right\} & \text{for } 1 \le s < s_1(q) \\ \exp\left\{-\beta N^2 \Phi_{II}(s)\right\} & \text{for } s_1(q) < s < s_2(q) \\ \exp\left\{-\beta N^{1+\frac{1}{q}}\ \Psi_{III}(s)\right\} & \text{for } s > s_2(q) \end{cases}$$



Computation of the pdf of Σ_q: Coulomb gas method

- PDF of $\Sigma_q = \sum_i \lambda_i^q$

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- The joint pdf of the eigenvalues can be interpreted as a **Boltzmann weight** at inverse temperature β:

$$\begin{aligned} P(\lambda_1, ..., \lambda_N) &\propto \delta\Big(\sum_i \lambda_i - 1\Big) \prod_{i=1}^{N} \lambda_i^{\frac{\beta}{2}(M-N+1)-1} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \\ &\propto \exp\left\{-\beta E\left[\{\lambda_i\}\right]\right\} \end{aligned}$$

where

$$E\left[\{\lambda_i\}\right] = -\gamma \sum_{i=1}^N \ln \lambda_i - \sum_{i<j} \ln |\lambda_i - \lambda_j| \quad \text{(with } \sum_i \lambda_i = 1\text{)}$$

$\longrightarrow$ effective energy of a 2D Coulomb gas of charges

Charge Density and Effective Energy

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Continuous **charge density** in the large N limit (regimes **I**, **II**):

$$\{\lambda_i\} \longrightarrow \rho(x) = \frac{1}{N} \sum_i \delta\left(x - \lambda_i N\right)$$

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with **effective energy**

$$\begin{aligned} E_s\left[\rho\right] &= -\frac{1}{2} \int_0^\infty \int_0^\infty dx\, dx'\, \rho(x)\rho(x')\, \ln|x - x'| \\ &+\mu_0 \left(\int_0^\infty dx\, \rho(x) - 1\right) \\ &+\mu_1 \left(\int_0^\infty dx\, x\, \rho(x) - 1\right) + \mu_2 \left(\int_0^\infty dx\, x^q\, \rho(x) - s\right) \end{aligned}$$

where μ_0, μ_1 and μ_2 are Lagrange multipliers

Saddle Point Solution

$$E_s[\rho] = -\frac{1}{2}\int_0^\infty \int_0^\infty dx dx' \, \rho(x)\rho(x') \ln|x - x'| + \mu_0 \left(\int_0^\infty dx \, \rho(x) - 1 \right)$$
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$V(x) \to$ **effective potential** for the charges.

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- Taking derivative with respect to x gives

$$\mathcal{P} \int_0^\infty dx' \, \frac{\rho_c(x')}{x - x'} = \mu_1 + q \, \mu_2 x^{q-1} = V'(x)$$

Steps for Computing the PDF of Σ_q

(i) find the **solution** $\rho_c(x)$ **of the integral equation**

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Tricomi's solution: density $\rho_c(x)$ with finite support $[L_1, L_2]$:

$$\rho_c(x) = \frac{1}{\pi\sqrt{x - L_1}\sqrt{L_2 - x}} \left[C - \mathcal{P} \int_{L_1}^{L_2} \frac{dy}{\pi} \frac{\sqrt{y - L_1}\sqrt{L_2 - y}}{x - y} \, V'(y) \right] ,$$

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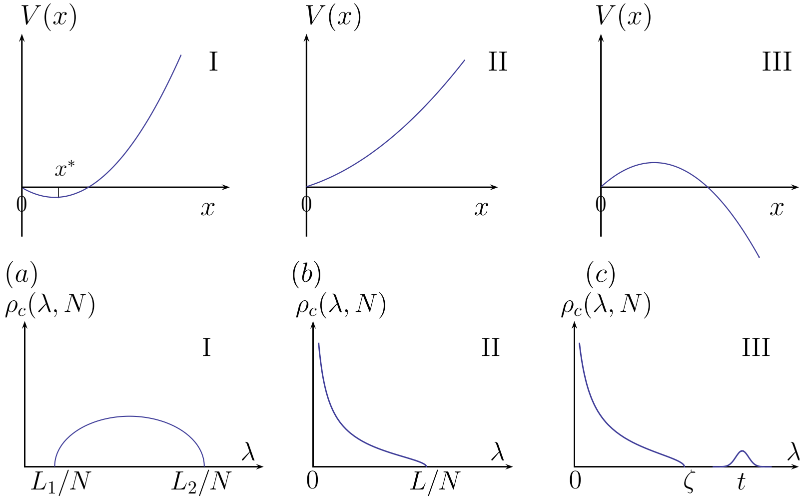
(iii) evaluate the **saddle point energy** $E_s[\rho_c]$

Phase Transitions

Regime I:
$1 < s < s_1$
$(s_1(q=2) = 5/4)$

Regime II:
$s_1 < s < s_2$
$(s_2(q=2) = 2 + \frac{2^{4/3}}{N^{1/3}})$

Regime III:
$s > s_2$
(weakly entangled)



$q = 2$: Purity

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$$\rho_c(x) = \frac{\sqrt{L_2 - x}\,\sqrt{x - L_1}}{2\pi\,(s-1)}\,,$$

$$P\left(\Sigma_2 = \frac{s}{N}, N\right) \propto e^{-\beta N^2 \Phi_I(s)}\,, \quad \Phi_I(s) = -\frac{1}{4}\ln\,(s-1) - \frac{1}{8}$$

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- **Regime II**: $5/4 < s < 2 + \frac{2^{4/3}}{N^{1/3}}$

$$\rho_c(x) = \frac{1}{\pi}\sqrt{\frac{L-x}{x}}\,(A + B\,x) \quad \text{with} \quad L = 2\left(3 - \sqrt{9-4s}\right)$$

$$P\left(\Sigma_2 = \frac{s}{N}, N\right) \propto e^{-\beta N^2 \Phi_{II}(s)}\,, \quad \Phi_{II}(s) = -\frac{1}{2}\ln\left(\frac{L}{4}\right) + \frac{6}{L^2} - \frac{5}{L} + \frac{7}{8}$$

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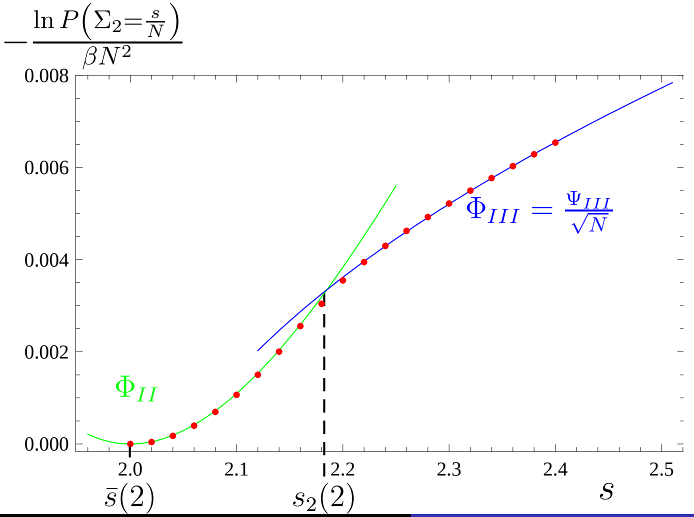
- **Regime III**: $s > 2 + \frac{2^{4/3}}{N^{1/3}}$ Continuous density with support $[0, \zeta]$ with $\zeta \approx \frac{4}{N}$ and separated $\lambda_{\max} \gg \zeta$: $\lambda_{\max} = t \approx \frac{\sqrt{s-2}}{\sqrt{N}}$

$$P\left(\Sigma_2 = \frac{s}{N}, N\right) \approx e^{-\beta N^{\frac{3}{2}}\,\Psi_{III}(s)}\,, \quad \Psi_{III}(s) = \frac{\sqrt{s-2}}{2}$$

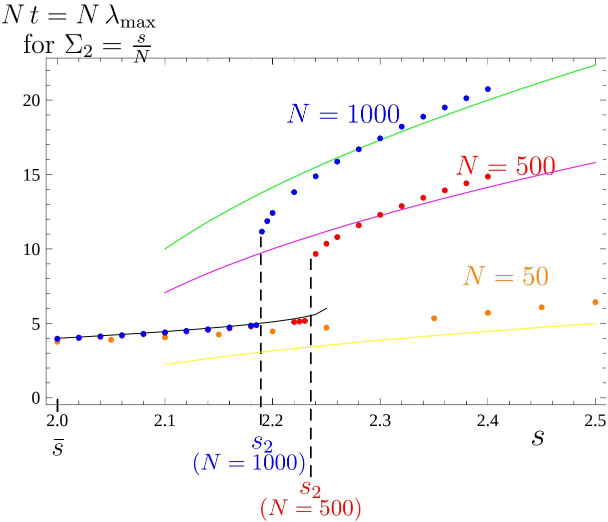
Numerical simulations

Monte Carlo Simulations (non-standard Metropolis algorithm):
pdf of Σ_2 (purity) for $N = 1000$ (second phase transition)



Numerical Simulations (2)

Jump of the maximal eigenvalue (rightmost charge) at $s = s_2$ for $q = 2$ and different values of N



The limit $q \to 1$ — von Neumann Entropy

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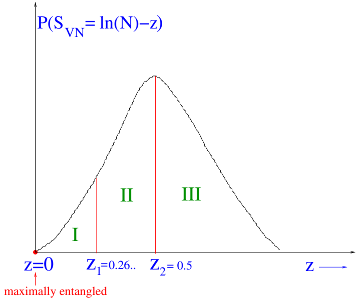

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- Around its mean, $\langle \lambda_{\max} \rangle = 4/N$, the typical fluctuations ($\sim O(N^{-5/3})$) are described by the Tracy-Widom distribution

$$\lambda_{\max} = \frac{4}{N} + 2^{4/3}\, N^{-5/3}\, Y_\beta$$

$Y_\beta \to$ random variable distributed via Tracy-Widom law

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- Around its mean, $\langle \lambda_{\max} \rangle = 4/N$, the typical fluctuations ($\sim O(N^{-5/3})$) are described by the Tracy-Widom distribution

$$\lambda_{\max} = \frac{4}{N} + 2^{4/3}\, N^{-5/3}\, Y_\beta$$

$Y_\beta \to$ random variable distributed via Tracy-Widom law

- For finite N and M, see P. Vivo (2011)

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Work in progress (C. Nadal, S.M with C. Pineda and T. Seligman)

Distribution of entropy for N finite but $M \gg N$ (large environment)?

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