Convex Hull of n Planar Brownian Motions

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Thanks to: D. Dhar (Tata Institute, Bombay, INDIA)

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Extended Review: arXiv: 0912:0631 (to appear in J. Stat. Phys. (2010))

Plan

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- Random Convex Hull $\Longrightarrow$ definition

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- Exact results for the mean perimeter and the mean area for all n.

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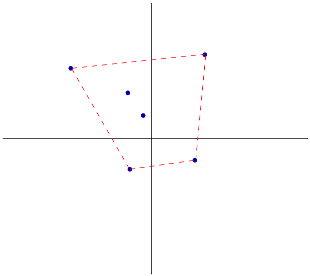
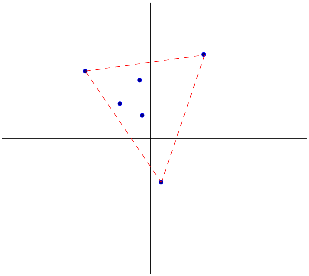
- Exact results for the mean perimeter and the mean area for all n.

- Summary and Conclusions

Random Convex Hull in a Plane





- Convex Hull $\Longrightarrow$ Minimal convex polygon enclosing the set

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- The shape of the convex hull $\rightarrow$ different for each sample

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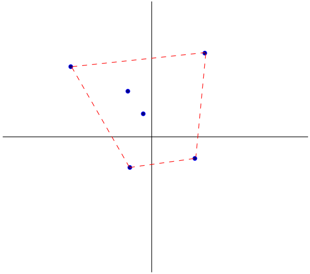

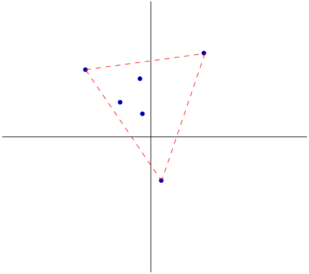

- Convex Hull $\Longrightarrow$ Minimal convex polygon enclosing the set
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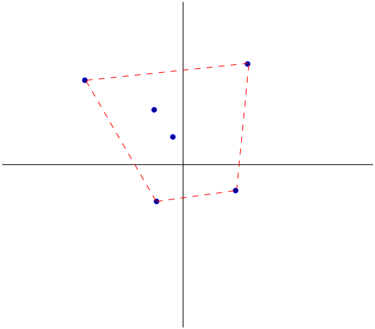
- Points drawn from a distribution $\rightarrow$ Independent or Correlated

Random Convex Hull in a Plane

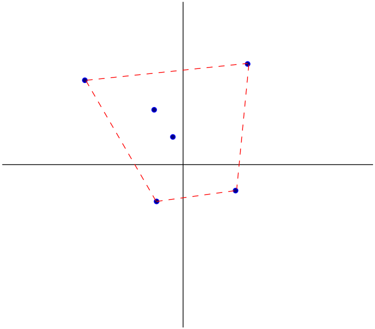
- Convex Hull $\Longrightarrow$ Minimal convex polygon enclosing the set
- The shape of the convex hull $\longrightarrow$ different for each sample
- Points drawn from a distribution $\longrightarrow$ Independent or Correlated
- Question: Statistics of observables: perimeter, area and no. of vertices

Independent Points in a Plane



Each point chosen independently from the same distribution

Independent Points in a Plane



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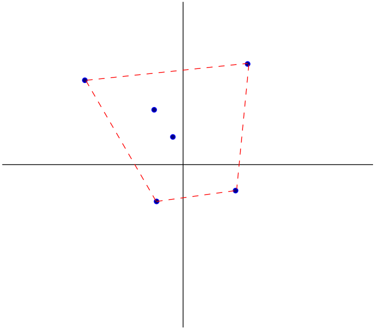
Associated Random Convex Hull $\longrightarrow$ well studied by diverse methods

Independent Points in a Plane



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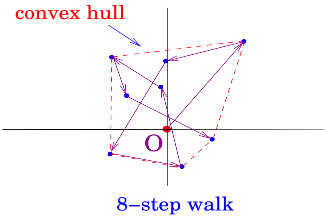
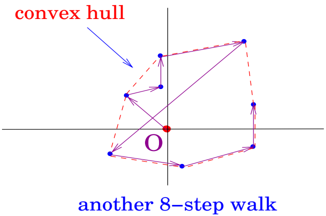
Associated Random Convex Hull $\longrightarrow$ well studied by diverse methods

P. Lévy ('48), J. Geffroy ('59), Spitzer & Widom ('59), Baxter ('59)..
Rényi & Sulanke ('63), Efron ('65), ...many others

Correlated Points: Vertices of an Open Random Walk





- Continuous-time limit: Brownian path of duration T

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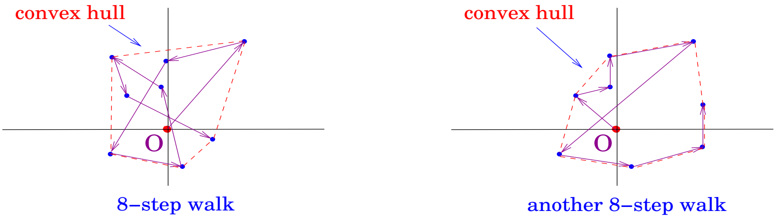
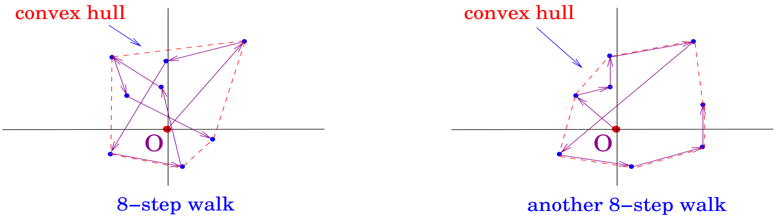


- Continuous-time limit: Brownian path of duration T
- mean perimeter and mean area of the associated Convex hull?

Correlated Points: Vertices of an Open Random Walk



- Continuous-time limit: Brownian path of duration T
- mean perimeter and mean area of the associated Convex hull?
- mean perimeter: $\langle L_1 \rangle = \sqrt{8\pi T}$ (Takács, '80)
- mean area: $\langle A_1 \rangle = \frac{\pi}{2}\, T$ (El Bachir, '83, Letac '93)

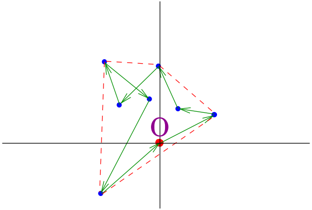
Correlated Points: Vertices of a Closed Random Walk

8 step random bridge

another 8 step bridge

• Continuous-time limit: Brownian bridge of duration T : starting at O and returning to it after time T

Correlated Points: Vertices of a Closed Random Walk



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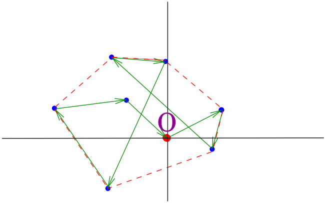

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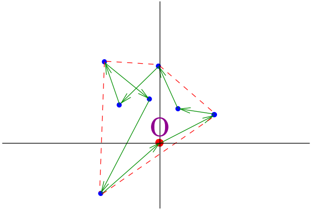

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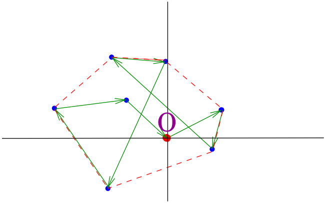

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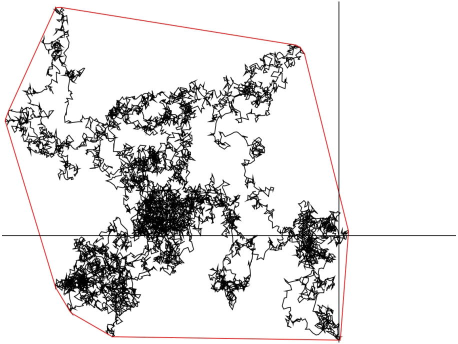
- Continuous-time limit: Brownian bridge of duration T : starting at O and returning to it after time T
- mean perimeter: $\langle L_1 \rangle = \sqrt{\frac{\pi^3}{2}\, T}$ (Goldman, '96).
- mean area: $\langle A_1 \rangle = ?$

Home Range Estimate via Convex Hull



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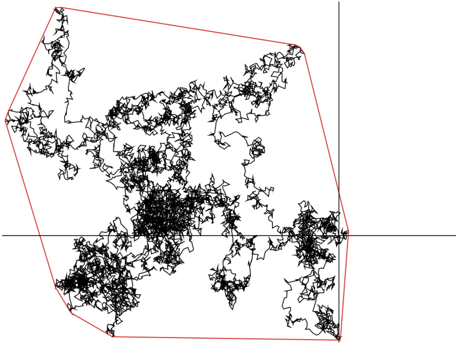

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Integrating Scientific Methods with Habitat Conservation Planning, Murphy and Noon (1992)
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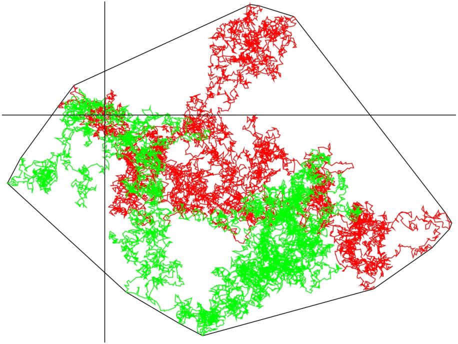

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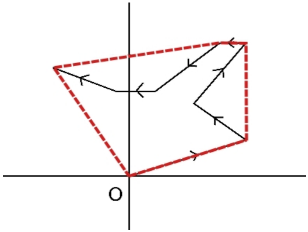
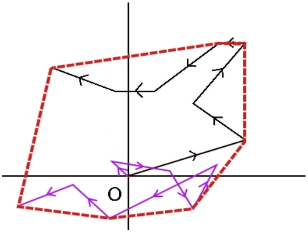
Global Convex Hull of n Independent Brownian Paths





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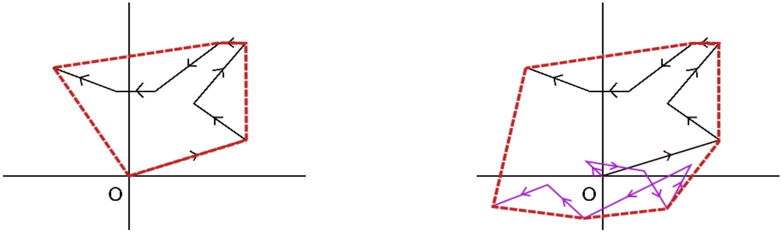

• Mean perimeter $\langle L_n \rangle$ and mean area $\langle A_n \rangle$ of n independent Brownian paths (bridges) each of duration T?

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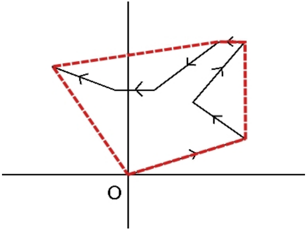

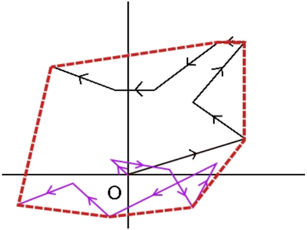

- Mean perimeter $\langle L_n \rangle$ and mean area $\langle A_n \rangle$ of n independent Brownian paths (bridges) each of duration T?
- $\langle L_n \rangle = \alpha_n \sqrt{T}$; $\qquad \langle A_n \rangle = \beta_n \, T$

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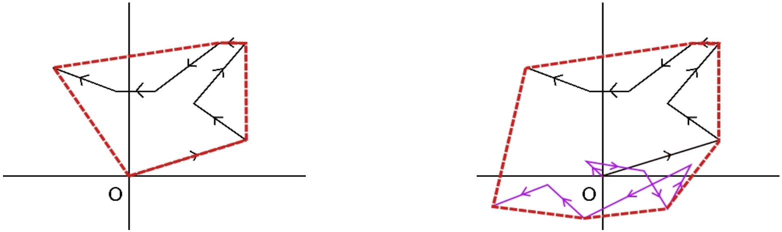
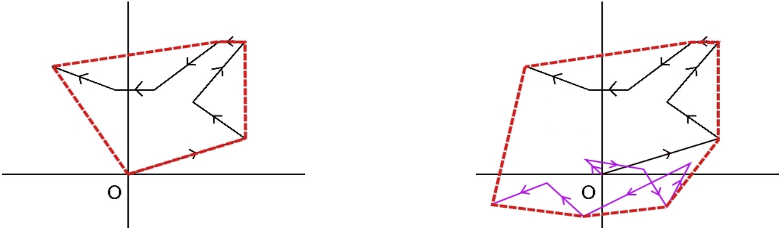


- Mean perimeter $\langle L_n \rangle$ and mean area $\langle A_n \rangle$ of n independent Brownian paths (bridges) each of duration T?
- $\langle L_n \rangle = \alpha_n \sqrt{T}$; $\qquad \langle A_n \rangle = \beta_n T$
- Recall $\boxed{\alpha_1 = \sqrt{8\pi}, \;\; \beta_1 = \pi/2}$ (open path)

  $\boxed{\alpha_1 = \sqrt{\pi^3/2}, \;\; \beta_1 = ?}$ (closed path)

Global Convex Hull of n Independent Brownian Paths



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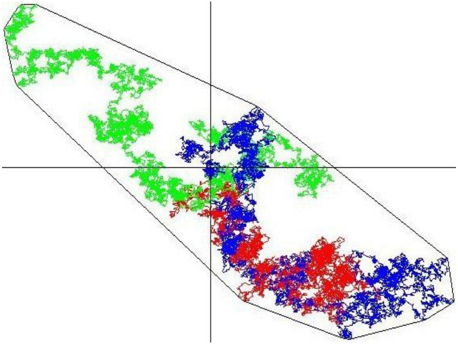
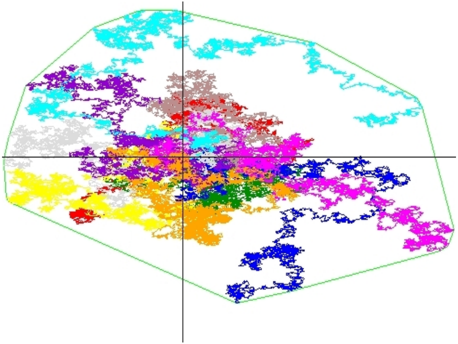
- $\boxed{\alpha_n, \;\; \beta_n = ?}$ $\rightarrow$ both for open and closed paths $\rightarrow$ n-dependence?

Global Convex Hull of n Independent Brownian Paths

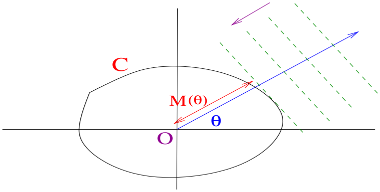


$n = 3$ closed paths



$n = 10$ open paths

Cauchy's Formulae for a Closed Convex Curve

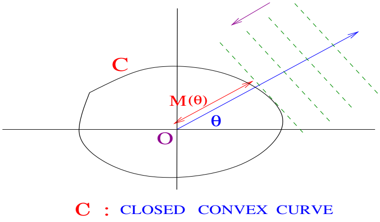


C : CLOSED CONVEX CURVE

- For any point $[X(s), Y(s)]$ on C define:

  Support function: $M(\theta) = \max_{s \in C} [X(s)\cos(\theta) + Y(s)\sin(\theta)]$

Cauchy's Formulae for a Closed Convex Curve



- For any point $[X(s), Y(s)]$ on C define:

  Support function: $M(\theta) = \max_{s \in C} [X(s)\cos(\theta) + Y(s)\sin(\theta)]$

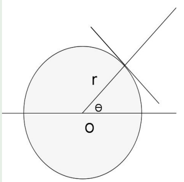
- Perimeter: $$L = \int_0^{2\pi} d\theta \; M(\theta)$$

- Area: $$A = \frac{1}{2}\int_0^{2\pi} d\theta \; \left[M^2(\theta) - [M'(\theta)]^2\right]$$

Examples

a circle centered at the origin:

$$M(\theta) = r$$



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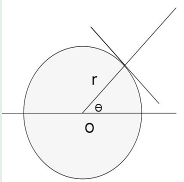

$$L = \int_0^{2\pi} d\theta \; M(\theta) = 2\pi r$$

$$A = \frac{1}{2} \int_0^{2\pi} d\theta \; \left[M^2(\theta) - [M'(\theta)]^2 \right] = \pi r^2$$

Examples

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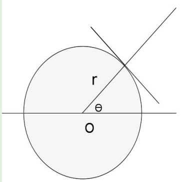
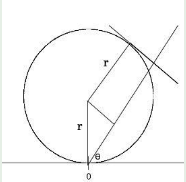


a circle touching the origin:

$$M(\theta) = r(1 + \sin\theta)$$



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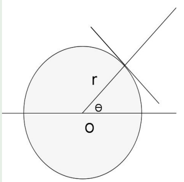

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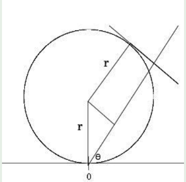

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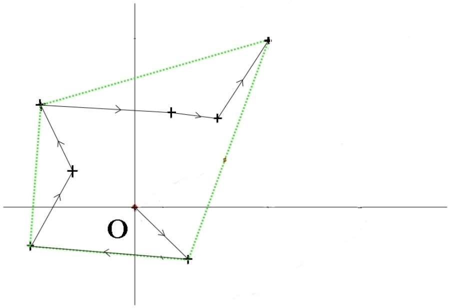
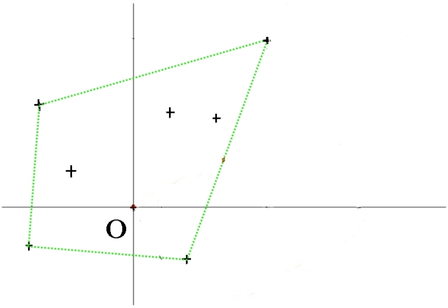
Cauchy's formulae Applied to Convex Polygon



Let $(x_k, y_k) \in I \Longrightarrow$ vertices of an N-step random walk starting at O

Let C (green) be the associated Convex Hull

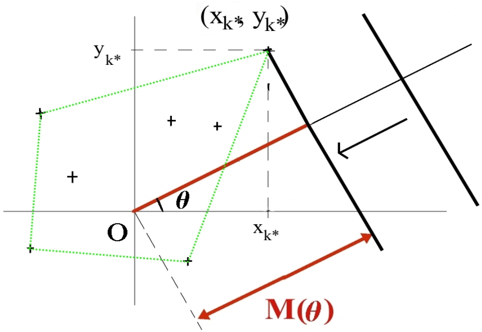
Cauchy's formulae Applied to Convex Polygon



$(x_k, y_k) \in I \implies$ vertices of the walk

$C \rightarrow$ Convex Hull with coordinates $\{X(s), Y(s)\}$ on C

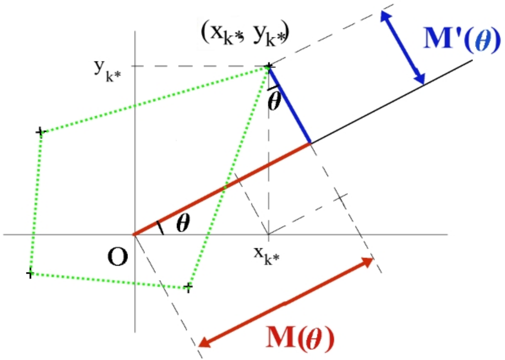
Cauchy's formulae Applied to Convex Polygon



$$M(\theta) = \max_{s \in C} [X(s)\cos\theta + Y(s)\sin\theta]$$
$$= \max_{k \in I} [x_k \cos\theta + y_k \sin\theta]$$
$$= x_{k^*}\cos\theta + y_{k^*}\sin\theta$$

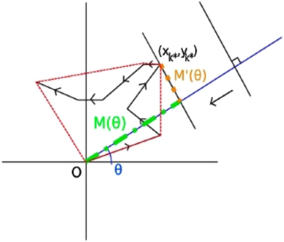
$k^* \rightarrow$ label of the point with largest projection along θ

Support Function of a Convex Hull



$$M(\theta) = x_{k^*} \cos\theta + y_{k^*} \sin\theta$$
$$M'(\theta) = -x_{k^*} \sin\theta + y_{k^*} \cos\theta$$

Cauchy's Formulae Applied to Random Convex Hull



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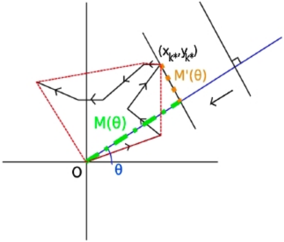


Mean perimeter of a random convex polygon

$$\langle L \rangle = \int_0^{2\pi} d\theta \; \langle M(\theta) \rangle$$

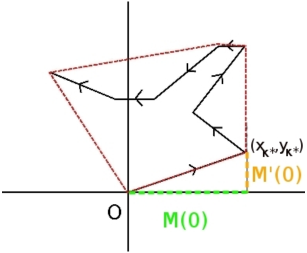
with $M(\theta) = x_{k^*} \cos\theta + y_{k^*} \sin\theta$

Mean area of a random convex polygon

$$\langle A \rangle = \frac{1}{2} \int_0^{2\pi} d\theta \left[\langle M^2(\theta) \rangle - \langle [M'(\theta)]^2 \rangle \right]$$

with $M'(\theta) = -x_{k^*} \sin\theta + y_{k^*} \cos\theta$

Isotropically Distributed Points

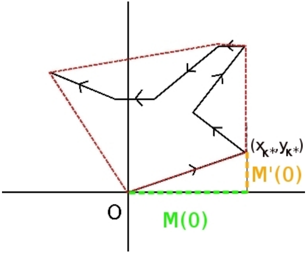


Mean Perimeter

$$\langle L \rangle = 2\pi \langle M(0) \rangle$$

with $M(\theta = 0) = \max_{k \in I} \{x_k\} = x_{k^*}$

Isotropically Distributed Points



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$$\langle A \rangle = \pi \left[\langle M^2(0) \rangle - \langle [M'(0)]^2 \rangle \right]$$

with $M'(\theta = 0) = y_{k^*}$

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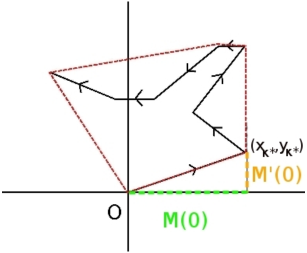

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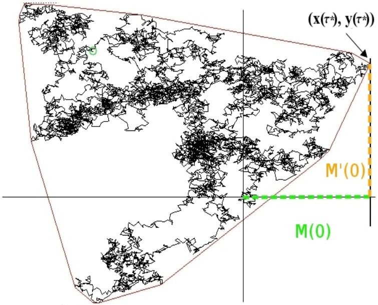
$\Longrightarrow$ Link to Extreme Value Statistics

Cauchy's Formulae Applied to the Convex Hull of a Brownian Path (n=1)



$x(\tau)$, $y(\tau)$ $\rightarrow$ a pair of independent one-dimensional Brownian motions: $0 \leq \tau \leq T$

$\frac{dx}{d\tau} = \eta_x(\tau)$
$\frac{dy}{d\tau} = \eta_y(\tau)$

Mean Perimeter

$$\langle L \rangle = 2\pi \langle M(0) \rangle$$

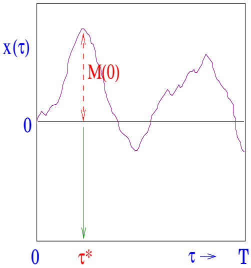
with $M(0) = \max_{0 \leq \tau \leq T} \{x(\tau)\} \equiv x(\tau^*)$

Mean Area

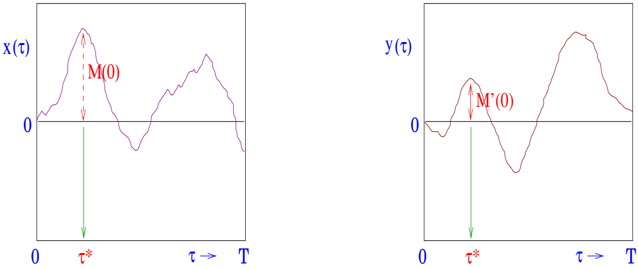
$$\langle A \rangle = \pi \left[\langle M^2(0) \rangle - \langle [M'(0)]^2 \rangle \right]$$

with $M'(0) = y(\tau^*)$

$M'(0) \to$ value of y at the special time τ^* when $x(\tau)$ is maximal

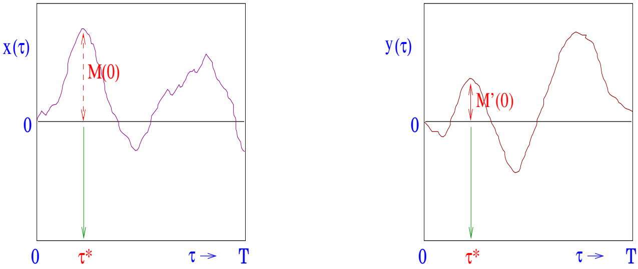


$M'(0) \to$ value of y at the special time τ^* when $x(\tau)$ is maximal



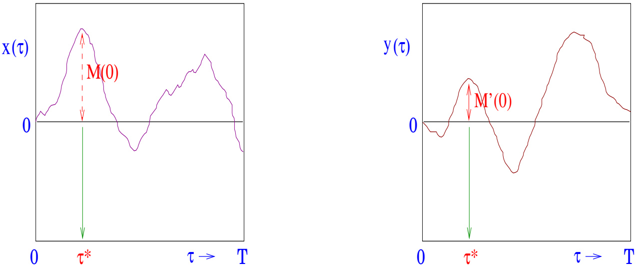
- $\langle M(0) \rangle = \int_0^\infty \, dM \, M \, \sigma_1(M|T)$; $\langle M^2(0) \rangle = \int_0^\infty \, dM \, M^2 \, \sigma_1(M|T)$

$M'(0) \to$ value of y at the special time τ^* when $x(\tau)$ is maximal



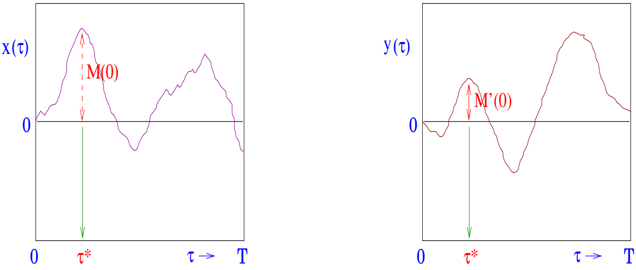
- $\langle M(0) \rangle = \int_0^\infty dM\ M\ \sigma_1(M|T)$; $\langle M^2(0) \rangle = \int_0^\infty dM\ M^2\ \sigma_1(M|T)$
- $\sigma_1(M|T) \to$ prob. density of maximum $M(0)$ of $x(\tau)$ in $[0, T]$

$M'(0) \to$ value of y at the special time τ^* when $x(\tau)$ is maximal



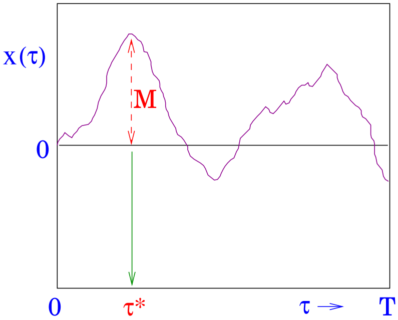
- $\langle M(0) \rangle = \int_0^\infty \; dM \; M \; \sigma_1(M|T)$; $\langle M^2(0) \rangle = \int_0^\infty \; dM \; M^2 \; \sigma_1(M|T)$
- $\sigma_1(M|T) \to$ prob. density of maximum $M(0)$ of $x(\tau)$ in $[0, T]$
- $\langle [M'(0)]^2 \rangle = \int_0^T \; d\tau^* \; \rho_1(\tau^*|T) \; \langle y^2(\tau^*) \rangle$

$M'(0) \to$ value of y at the special time τ^* when $x(\tau)$ is maximal

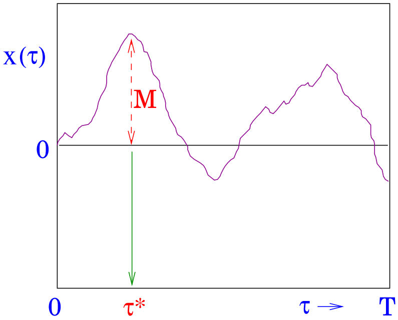


- $\langle M(0)\rangle = \int_0^\infty \; dM \; M \; \sigma_1(M|T)$; $\langle M^2(0)\rangle = \int_0^\infty \; dM \; M^2 \; \sigma_1(M|T)$
- $\sigma_1(M|T) \to$ prob. density of maximum $M(0)$ of $x(\tau)$ in $[0, T]$
- $\langle [M'(0)]^2 \rangle = \int_0^T \; d\tau^* \; \rho_1(\tau^*|T) \; \langle y^2(\tau^*)\rangle = \langle \tau^* \rangle$

Distribution of M and τ^* for a single Brownian Path

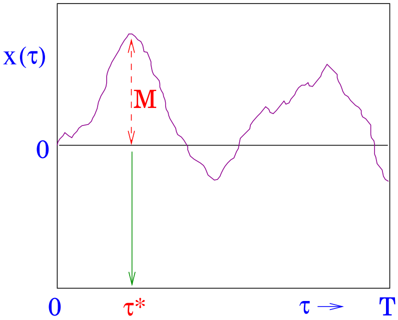


Distribution of M and τ^* for a single Brownian Path



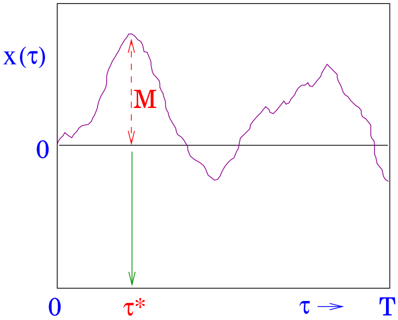
- Joint Distribution: $P_1(M, \tau^* | T) = \frac{M}{\pi \tau^{*3/2} \sqrt{T - \tau^*}} \, e^{-M^2/2\tau^*}$

Distribution of M and τ^* for a single Brownian Path



- Joint Distribution: $P_1(M,\tau^*|T) = \frac{M}{\pi {\tau^*}^{3/2}\sqrt{T-\tau^*}}\, e^{-M^2/2\tau^*}$
- Marginals: $\sigma_1(M|T) = \sqrt{\frac{2}{\pi T}}\, e^{-M^2/2T}$

Distribution of M and τ^* for a single Brownian Path



- Joint Distribution: $P_1(M, \tau^*|T) = \frac{M}{\pi \tau^{*3/2}\sqrt{T-\tau^*}}\, e^{-M^2/2\tau^*}$

- Marginals: $\sigma_1(M|T) = \sqrt{\frac{2}{\pi T}}\, e^{-M^2/2T}$

$$\rho_1(\tau^*|T) = \frac{1}{\pi\sqrt{\tau^*(T-\tau^*)}} \longrightarrow \text{Lévy's arcsine law}$$

Distribution of the time τ^* at which a Brownian Motion is maximal over [0,T]

Lévy's Arcsine Law: $\rho_1(\tau^*|T) = \frac{1}{T}\ f_1\left(\frac{\tau^*}{T}\right)$

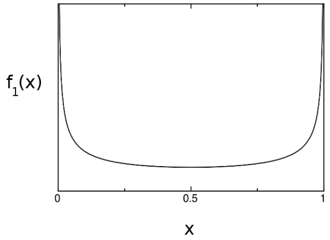
$$f_1(x) = \frac{1}{\pi\sqrt{x(1-x)}}$$

Distribution of the time τ^* at which a Brownian Motion is maximal over [0,T]

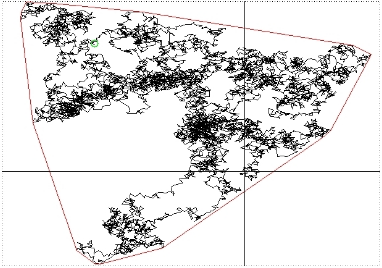
Lévy's Arcsine Law: $\rho_1(\tau^*|T) = \frac{1}{T}\ f_1\left(\frac{\tau^*}{T}\right)$

$$f_1(x) = \frac{1}{\pi\sqrt{x(1-x)}}$$

Cumulative distribution: $\mathrm{Prob}(\tau^* \leq t|T) = \frac{2}{\pi}\,\mathrm{arcsin}\left(\sqrt{t}\right)$



Results for n=1 Open Brownian Path



$x(\tau)$, $y(\tau)$ $\to$ a pair of independent one-dimensional Brownian motions over $0 \leq \tau \leq T$

Mean Perimeter

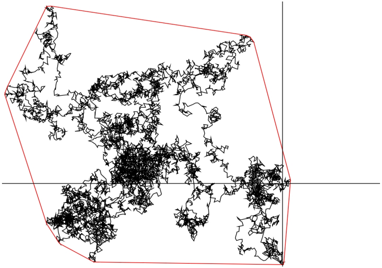
$$\langle L \rangle = \sqrt{8\pi T}$$

Mean Area

$$\langle A \rangle = \frac{\pi T}{2}$$

Takács, *Expected perimeter length*, Amer. Math. Month., **87** (1980)
El Bachir, (1983)

Results for n=1 Closed Brownian Path

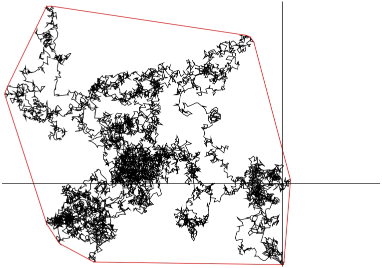


$x(\tau)$, $y(\tau)$ $\rightarrow$ a pair of independent one-dimensional Brownian bridges over $0 \leq \tau \leq T$

Mean Perimeter

$\langle L \rangle = \sqrt{\frac{\pi^3 T}{2}}$ Goldman, '96

Results for n=1 Closed Brownian Path



$x(\tau)$, $y(\tau)$ $\to$ a pair of independent one-dimensional Brownian bridges over $0 \leq \tau \leq T$

Mean Perimeter

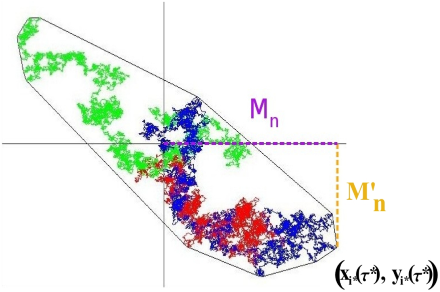
$\langle L \rangle = \sqrt{\frac{\pi^3 T}{2}}$ Goldman, '96

Mean Area

$$\boxed{\langle A \rangle = \frac{\pi T}{3}}$$

$\to$ New Result

Convex Hull of n Independent Brownian Paths



$x_i(\tau)$, $y_i(\tau) \to 2\,n$ independent one-dimensional Brownian paths each of duration T

Mean Perimeter

$$\langle L_n \rangle = 2\pi \langle M_n \rangle$$

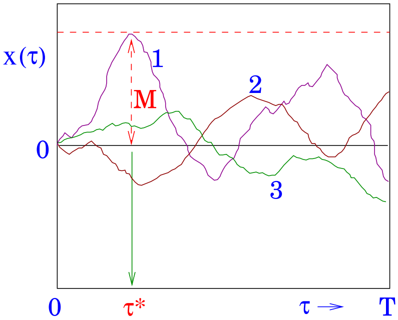
with $M_n = \max_{\tau,i} \{x_i(\tau)\} \equiv x_{i^*}(\tau^*)$

Mean Area

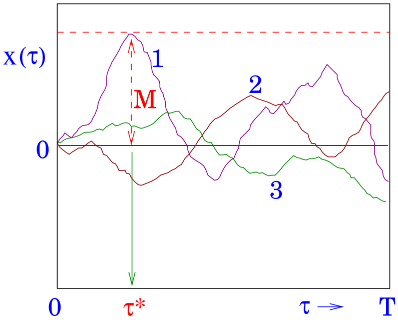
$$\langle A_n \rangle = \pi \left[\langle M_n^2 \rangle - \langle [M_n']^2 \rangle \right]$$

with $M_n' = y_{i^*}(\tau^*)$

Distribution of the global maximum M and τ^* for n paths



Distribution of the global maximum M and τ^* for n paths



- Joint Distribution: $P_n(M,\tau^*|T) = n\,P_1(M,\tau^*|T)\,\left[\mathrm{erf}\left(\frac{M}{\sqrt{2T}}\right)\right]^{n-1}$

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \, du\ e^{-u^2}$$

Marginals of M and τ^* for arbitrary n

- Marginals: $\sigma_n(M|T) = \sqrt{\frac{2}{\pi T}}\, n\, e^{-M^2/2T} \left[\mathrm{erf}\left(\frac{M}{\sqrt{2T}}\right)\right]^{n-1}$

Marginals of M and τ^* for arbitrary n

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$$\rho_n(\tau^*|T) = \frac{1}{T}\, f_n\left(\tau^*/T\right)$$

Marginals of M and τ^* for arbitrary n

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$$\rho_n(\tau^*|T) = \frac{1}{T}\, f_n\,(\tau^*/T)$$

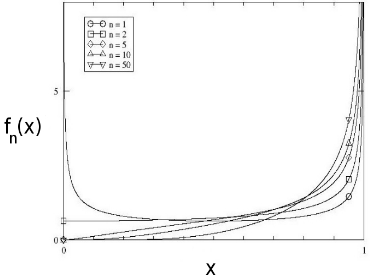
$$f_n(x) = \frac{2n}{\pi\sqrt{x(1-x)}} \int_0^\infty u\; e^{-u^2}\; [\mathrm{erf}(u\sqrt{x})]^{n-1}\; du$$

Marginals of M and τ^* for arbitrary n

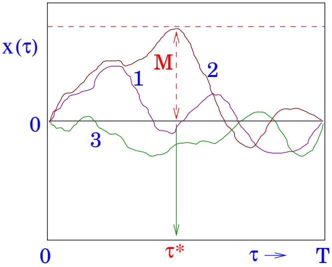
- Marginals: $\sigma_n(M|T) = \sqrt{\frac{2}{\pi T}}\, n\, e^{-M^2/2T} \left[\mathrm{erf}\left(\frac{M}{\sqrt{2T}}\right)\right]^{n-1}$

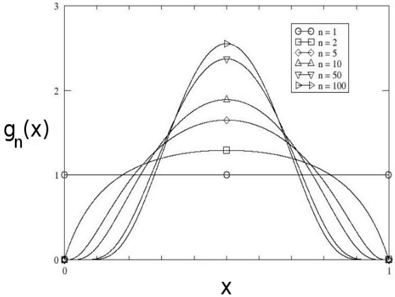
$$\rho_n(\tau^*|T) = \frac{1}{T} f_n(\tau^*/T)$$

$$f_n(x) = \frac{2n}{\pi\sqrt{x(1-x)}} \int_0^\infty u\ e^{-u^2}\ [\mathrm{erf}(u\sqrt{x})]^{n-1}\ du$$

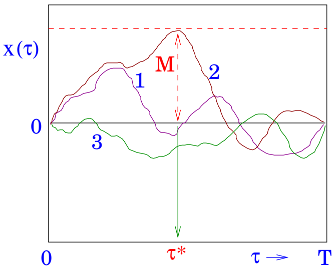


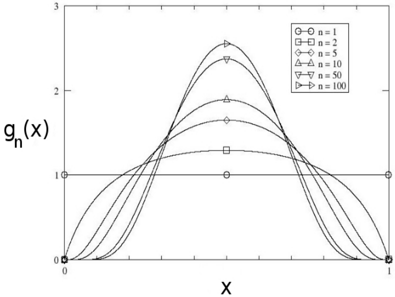
Marginals for n Independent Brownian Bridges





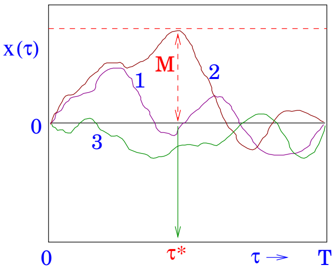
Marginals for n Independent Brownian Bridges

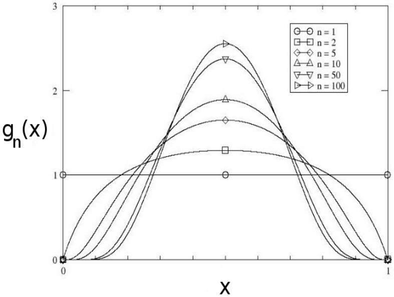




- Marginals: $\sigma_n(M|T) = \frac{4n}{T}\, M \left(1 - e^{-2M^2/T}\right)^{n-1}$

Marginals for n Independent Brownian Bridges

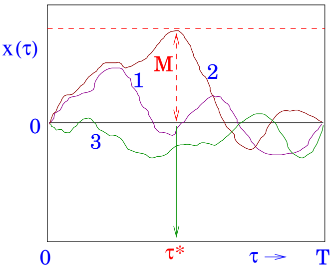


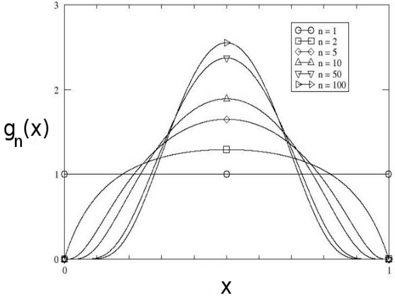


- Marginals: $\sigma_n(M|T) = \frac{4n}{T}\, M\, \left(1 - e^{-2M^2/T}\right)^{n-1}$

  $\rho_n(\tau^*|T) = \frac{1}{T}\, g_n\,(\tau^*/T)$

Marginals for n Independent Brownian Bridges



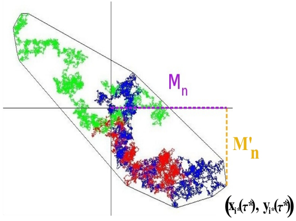


- Marginals: $\sigma_n(M|T) = \frac{4n}{T}\, M \left(1 - e^{-2M^2/T}\right)^{n-1}$

$$\rho_n(\tau^*|T) = \frac{1}{T}\, g_n\left(\tau^*/T\right)$$

$$g_n(x) = n \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(-1)^k}{\left[1 + 4\, k\, x\, (1-x)\right]^{\frac{3}{2}}}$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Open Brownian Paths



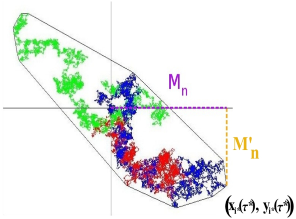
$x_i(\tau)$, $y_i(\tau) \to 2\,n$ independent one-dimensional Brownian paths over $0 \le \tau \le T$

Mean Perimeter (open paths)

$$\langle L_n \rangle = 2\pi \langle M_n \rangle$$

with $M_n = \max_{\tau,i} \{x_i(\tau)\} \equiv x_{i^*}(\tau^*)$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Open Brownian Paths



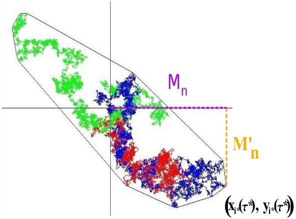
$x_i(\tau)$, $y_i(\tau)$ $\to$ $2\,n$ independent one-dimensional Brownian paths over $0 \le \tau \le T$

Mean Perimeter (open paths)

$$\langle L_n \rangle = \alpha_n \sqrt{T}$$

$$\alpha_n = 4n\sqrt{2\pi} \int_0^\infty du\ u\ e^{-u^2}\ [\mathrm{erf}(u)]^{n-1}$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Open Brownian Paths



$x_i(\tau)$, $y_i(\tau) \rightarrow 2\,n$ independent one-dimensional Brownian paths over $0 \leq \tau \leq T$

Mean Perimeter (open paths)

$$\langle L_n \rangle = \alpha_n \sqrt{T}$$

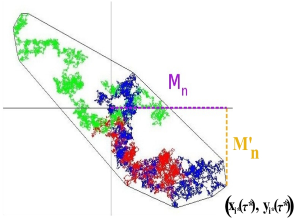
$$\alpha_n = 4n\sqrt{2\pi} \int_0^\infty du\ u\ e^{-u^2}\ [\mathrm{erf}(u)]^{n-1}$$

$$\alpha_1 = \sqrt{8\pi} = 5,013..$$

$$\alpha_2 = 4\sqrt{\pi} = 7,089..$$

$$\alpha_3 = 24\,\frac{\tan^{-1}\left(1/\sqrt{2}\right)}{\sqrt{\pi}} = 8,333..$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Open Brownian Paths



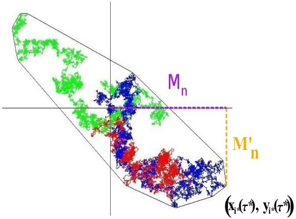
$x_i(\tau)$, $y_i(\tau) \to 2\,n$ independent one-dimensional Brownian paths over $0 \le \tau \le T$

Mean Area (open paths)

$$\langle A_n \rangle = \pi \left[\langle M_n^2 \rangle - \langle [M_n']^2 \rangle \right]$$

with $M_n' = y_{i^*}(\tau^*)$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Open Brownian Paths



$x_i(\tau)$, $y_i(\tau) \to 2\,n$ independent one-dimensional Brownian paths over $0 \leq \tau \leq T$

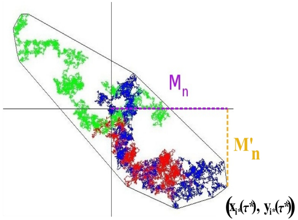
Mean Area (open paths)

$\langle A_n \rangle = \beta_n T$

$$\beta_n = 4n\sqrt{\pi}\int_0^\infty du\; u\; [\mathrm{erf}(u)]^{n-1}\left(ue^{-u^2} - h(u)\right)$$

$$h(u) = \frac{1}{2\sqrt{\pi}}\int_0^1 \frac{e^{-u^2/t}\;dt}{\sqrt{t(1-t)}}$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Open Brownian Paths



$x_i(\tau)$, $y_i(\tau)$ $\rightarrow$ $2\,n$ independent one-dimensional Brownian paths over $0 \leq \tau \leq T$

Mean Area (open paths)

$$\langle A_n \rangle = \beta_n T$$

$$\beta_n = 4n\sqrt{\pi} \int_0^\infty du\ u\ [\mathrm{erf}(u)]^{n-1} \left(u e^{-u^2} - h(u) \right)$$

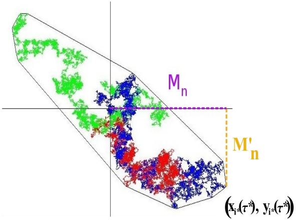
$$h(u) = \frac{1}{2\sqrt{\pi}} \int_0^1 \frac{e^{-u^2/t}\ dt}{\sqrt{t(1-t)}}$$

$$\beta_1 = \frac{\pi}{2} = 1,570..$$

$$\beta_2 = \pi = 3,141..$$

$$\beta_3 = \pi + 3 - \sqrt{3} = 4,409..$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Closed Brownian Paths



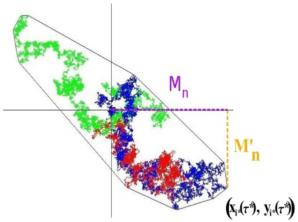
$x_i(\tau)$, $y_i(\tau) \rightarrow 2\,n$ independent one-dimensional Brownian bridges over $0 \leq \tau \leq T$

Mean Perimeter (Closed Paths)

$\langle L_n^c \rangle = \alpha_n^c \sqrt{T}$

$$\alpha_n^c = \frac{\pi^{3/2}}{\sqrt{2}} \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{\sqrt{k}}$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Closed Brownian Paths



$x_i(\tau)$, $y_i(\tau) \to 2\,n$ independent one-dimensional Brownian bridges over $0 \le \tau \le T$

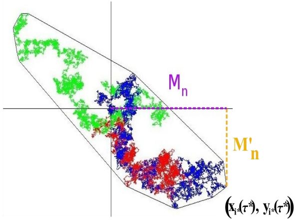
Mean Perimeter (Closed Paths)

$\langle L_n^c \rangle = \alpha_n^c \sqrt{T}$

$$\alpha_n^c = \frac{\pi^{3/2}}{\sqrt{2}} \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{\sqrt{k}}$$

$$\begin{aligned}
\alpha_1^c &= \sqrt{\pi^3/2} = 3,937. \\
\alpha_2^c &= \sqrt{\pi^3}(\sqrt{2} - 1/2) = 5,090.. \\
\alpha_3^c &= \sqrt{\pi^3}\left(\frac{3}{\sqrt{2}} - \frac{3}{2} + \frac{1}{\sqrt{6}}\right) = 5,732..
\end{aligned}$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Closed Brownian Paths



$x_i(\tau)$, $y_i(\tau) \to 2\,n$ independent one-dimensional Brownian bridges over $0 \le \tau \le T$

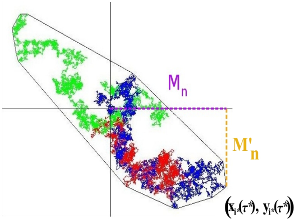
Mean Area (Closed Paths)

$\langle A_n^c \rangle = \beta_n^c\, T$

$$\beta_n^c = \frac{\pi}{2}\left[\sum_{k=1}^{n}\frac{1}{k} - \frac{n}{3} + \frac{1}{2}\sum_{k=2}^{n}(-1)^k\, w(k)\right]$$

$$w(k) = \binom{n}{k}\,(k-1)^{-3/2}\left(k\tan^{-1}(\sqrt{k-1}) - \sqrt{k-1}\right)$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Closed Brownian Paths



$x_i(\tau),\ y_i(\tau) \to 2\,n$ independent one-dimensional Brownian bridges over $0 \le \tau \le T$

Mean Area (Closed Paths)

$\langle A_n^c \rangle = \beta_n^c\, T$

$$\beta_n^c = \frac{\pi}{2}\left[\sum_{k=1}^{n}\frac{1}{k} - \frac{n}{3} + \frac{1}{2}\sum_{k=2}^{n}(-1)^k\, w(k)\right]$$

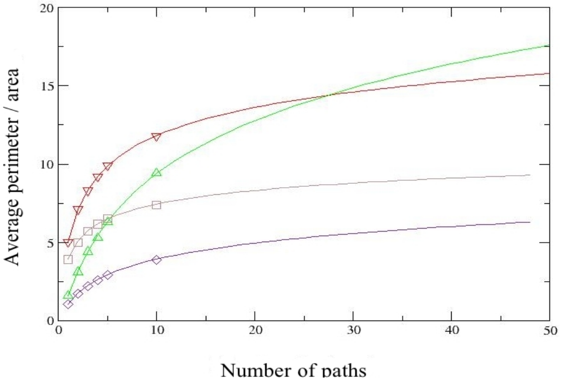
$$w(k) = \binom{n}{k}\,(k-1)^{-3/2}\,\left(k\tan^{-1}(\sqrt{k-1}) - \sqrt{k-1}\right)$$

$$\beta_1^c = \frac{\pi}{3} = 1,047..$$

$$\beta_2^c = \frac{\pi(4+3\pi)}{24} = 1,757..$$

$$\beta_3^c = 2,250..$$

Numerical Check



The coefficients α_n (mean perimeter) (lower triangle), β_n (mean area) (upper triangle) of n open paths and similarly α_n^c (square) and β_n^c (diamond) for n closed paths, plotted against n. The symbols denote numerical simulations (up to $n = 10$, with 10^3 realisations for each point)

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- Unified approach adapting Cauchy's formulae

  $\Longrightarrow$ Mean Perimeter and Area of Random Convex Hull

  both for Independent and Correlated points

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  $\Longrightarrow$ Ecological Implication: Home Range Estimate

Very slow (logarithmic) growth of Home Range with population size n

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- 3 dimensions: Random Convex Polytopes ?