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Fractional Calculus from mathematician's paradox to application in physics and engineering

Shantanu Das

Scientist, RCSDS E&I Group BARC Mumbai,

Senior Research Professor Dept. of Phys, Jadavpur Univ (JU),

Adjunct Professor DIAT-Pune

UGC-Visiting Fellow Dept. of Appl. Math Calcutta Univ.

shantanu@barc.gov.in

http://scholar.google.co.uk/citations?user=9ix9YS8AAAAJ&hl=en

www.shantanudaslecture.com

From Laplace operator the fractional derivative

Classically we have $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ where $\mathcal{L}\{f(t)\} = F(s)$

Say for case of RHS if we have $s^{\alpha}F(s) - f(0)$ for $0 < \alpha < 1$

Then we relate to $\mathcal{L}\{f^{(\alpha)}(t)\} = s^{\alpha}F(s) - f(0)$ as $\alpha-$ th derivative of $f(t)$

For α - negative we have $\alpha-$ order integration i.e. with $\alpha = -\beta$

$$\mathcal{L}\{f^{(-\beta)}(t)\} = s^{-\beta}F(s)$$

So there is possibility that we have in between operations for integration & differentiation, like half, one third etc $D^{1/2} f(t)$ $D^{1/3} f(t)$ $D^{1\frac{1}{2}} f(t)$ $D^{-1/2} f(t)$

We are having thus fractional Laplace variables like $s^{1/2}$ $s^{-1/2}$ etc

Assume presently that we have fractional differentiation and integration then how can we use the corresponding fractional Laplace variable s^{α} ?

CFE for approximation of fractional semi differential Laplace operator

CFE is Continued Fraction Expansion

CFE is defined as following

$$(1+x)^{\alpha} \overset{\text{def}}{=} \cfrac{1}{1-\alpha \cfrac{x}{1+\left(\frac{1}{2}\right)(\alpha+1)\cfrac{x}{1-\left(\frac{1}{6}\right)(\alpha-1)\cfrac{x}{1+\left(\frac{1}{6}\right)(\alpha+2)\cfrac{x}{1-\left(\frac{1}{10}\right)(\alpha-2)\cfrac{x}{1+\ldots}}}}}}$$

For obtaining rational approximation of $\sqrt{s}$ put in CFE $x = s - 1$ and $\alpha = \frac{1}{2}$

CFE for four number term approximation for $\sqrt{s}$ is $\sqrt{s} \approx \dfrac{5s^2 + 10s + 1}{s^2 + 10s + 5}$

Here we have got approximation for half-differentiation in Laplace domain-is implementable

We can have a transfer function of a filter to realize fractional Laplace variable

CFE for approximation of fractional semi differential Laplace Operator for various number of terms

S.No	No. of terms in CFE approximation	Rational approximation for $\sqrt{s}$
1	2	$\frac{3s+1}{s+3}$
2	4	$\frac{5s^2+10s+1}{s^2+10s+5}$
3	6	$\frac{7s^3+35s^2+21s+1}{s^3+21s^2+35s+7}$
4	8	$\frac{11s^5+165s^4+462s^3+330s^2+55s+1}{s^5+55s^4+330s^3+462s^2+165s+11}$

CFE gives approximation for fractional Laplace operator in terms of ratio of rational polynomials.

The above is semi-differentiation operator. The semi-integration operator will be reciprocal of the above ratio.

Makes way to have fractional order analog and digital circuits & systems

We make use of the fractional Laplace variables to make controller

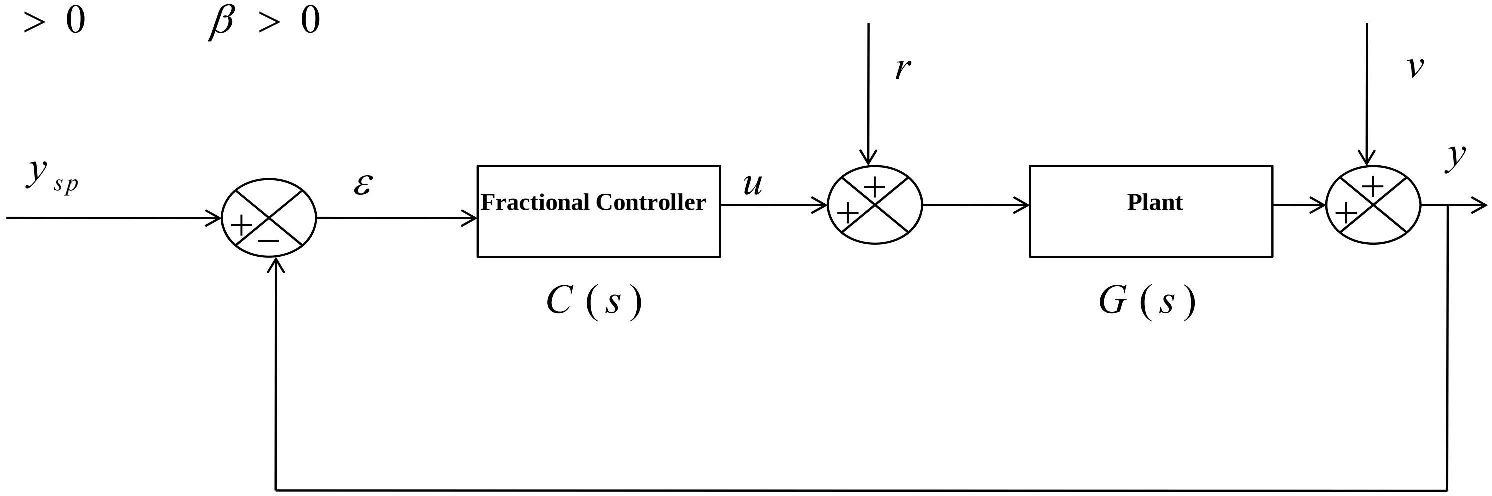
$$C_{PID}(s) = k_p + \frac{k_i}{s} + k_d s$$

$$u_{PID}(t) = k_p \varepsilon(t) + k_i D^{-1} \varepsilon(t) + k_d D^{1} \varepsilon(t)$$

$$C_{FOPID}(s) = k_p + \frac{k_i}{s^{\alpha}} + k_d s^{\beta}$$

$$u_{FOPID}(t) = k_p \varepsilon(t) + k_i D^{-\alpha} \varepsilon(t) + k_d D^{\beta} \varepsilon(t)$$

$\alpha > 0 \qquad \beta > 0$



y_{sp} : **Set-point**

ε : **Error**

u : **Controller output**

r : **Load disturbance**

v : **Measurement Noise**

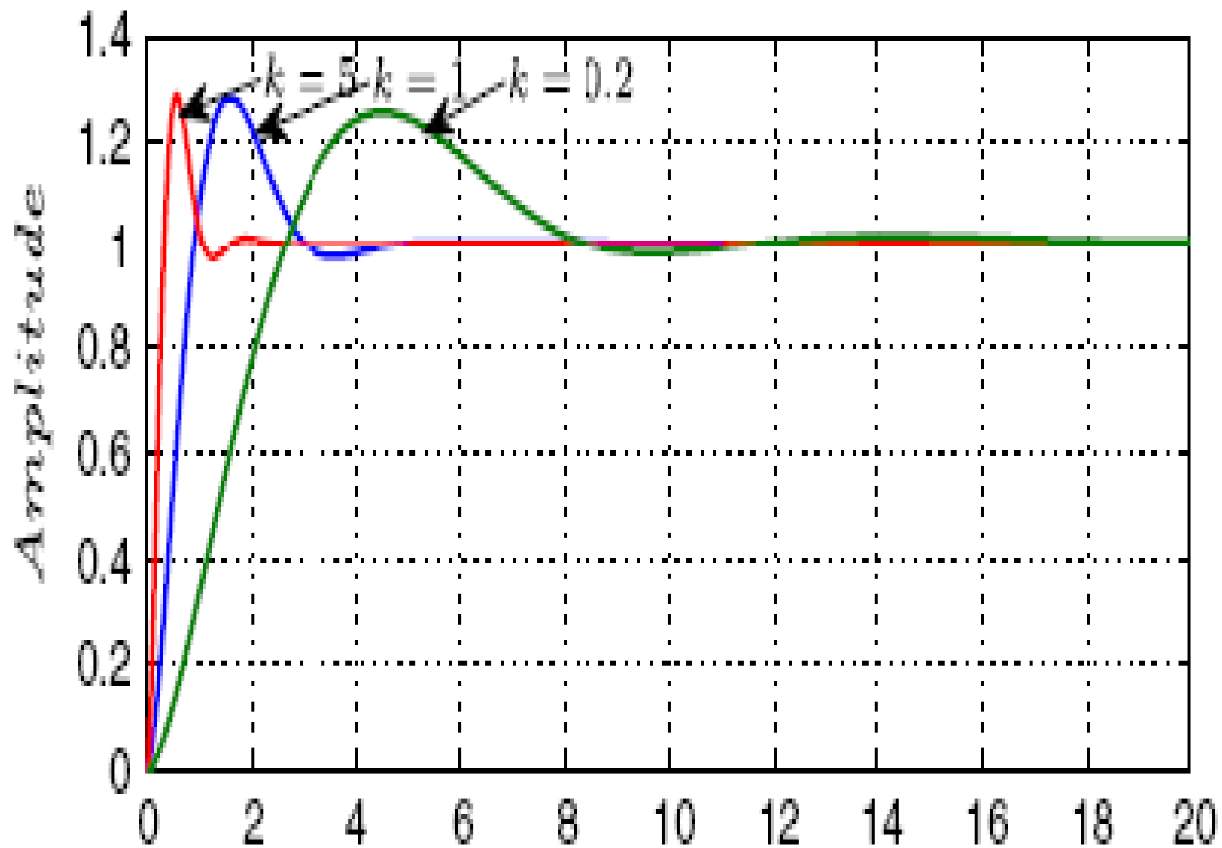
y : **Out-put**

We have a fractional order PID (FO-PID)system called as $PI^{\alpha}D^{\beta}$

Observe robust control

We observe iso-damping for uncertain plant $G(s)$

Plant with gain uncertainty $$G(s) = \frac{k}{s^2 + 2\xi\omega_0 s + s^2}$$ $k \in [0.2, 5.0]$

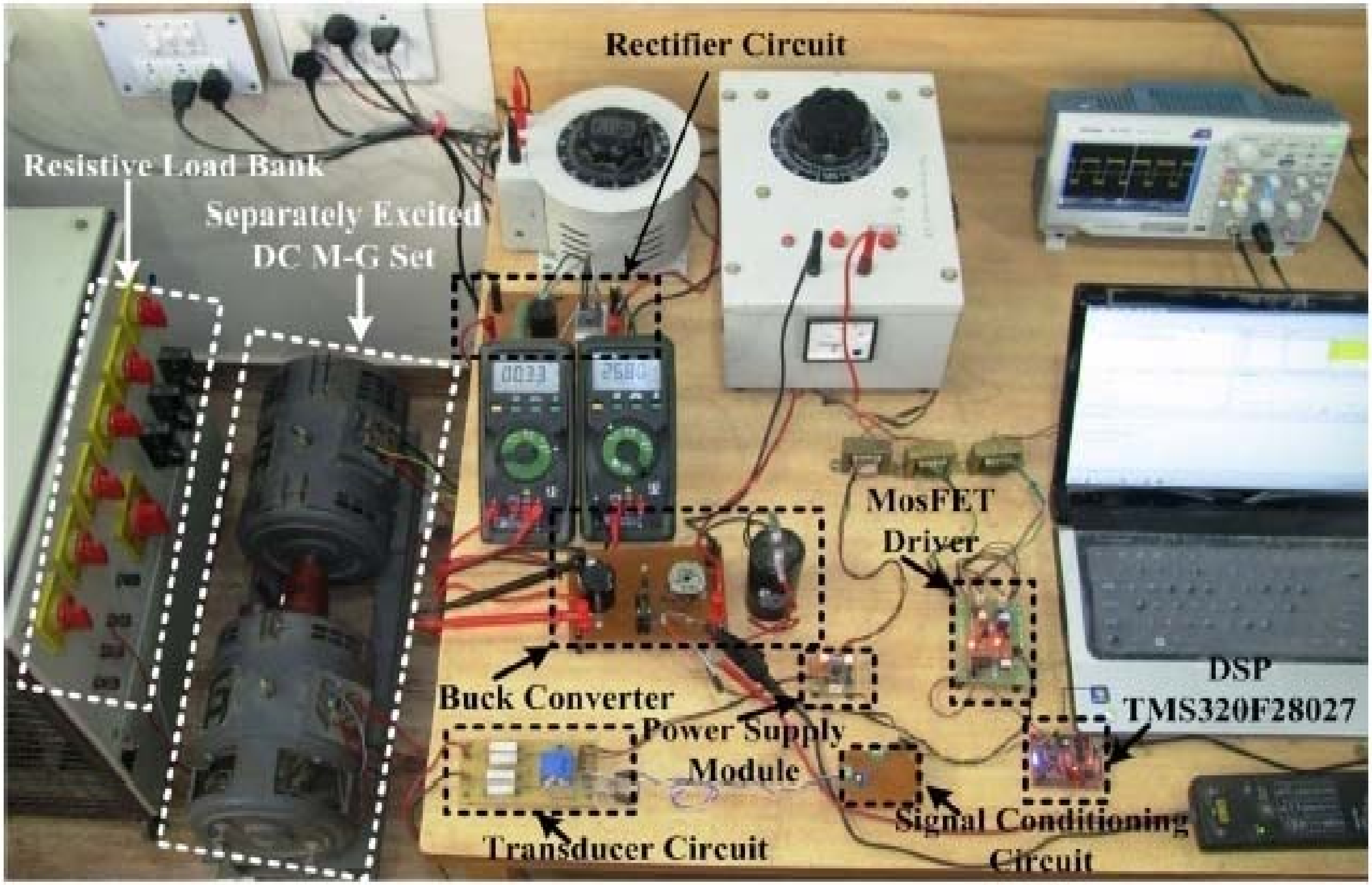


$y(t)$

t

Enhanced robustness

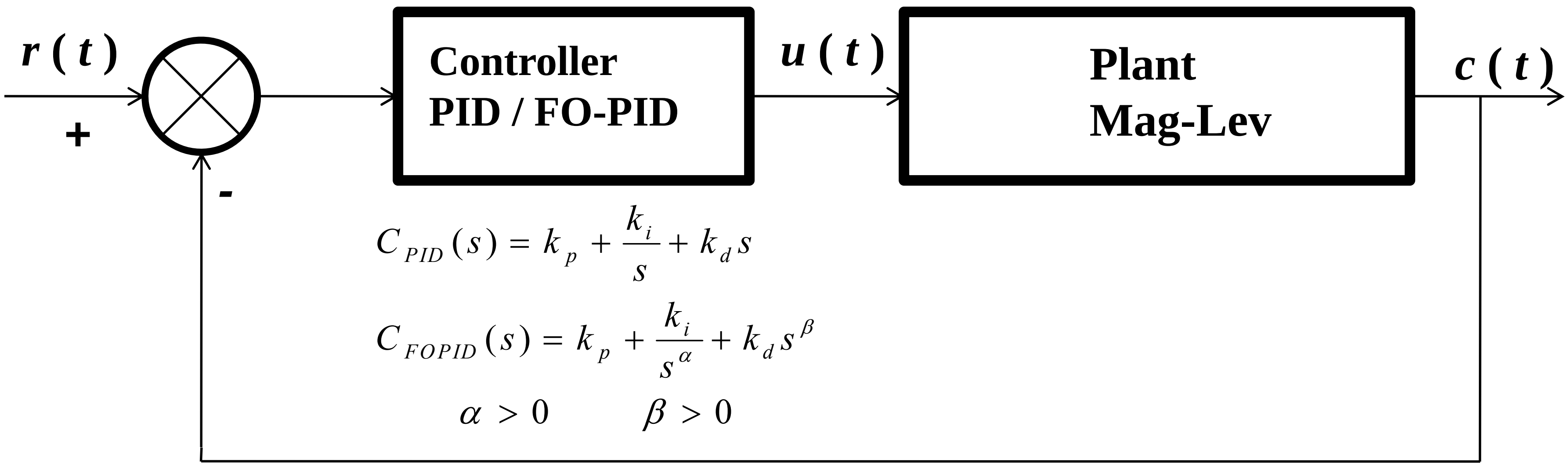
Digital FO-PID for DC Motor Speed Control hardware



Circuit Systems & Process Springer DOI 10.1007/s00034-016-0262-2:2016

Courtesy: BRNS funded joint project of VNIT Nagpur and BARC to develop industrial digital fractional order PID.

Controller output $u(t)$ for PID and Fractional PID



$$C_{PID}(s) = k_p + \frac{k_i}{s} + k_d s$$

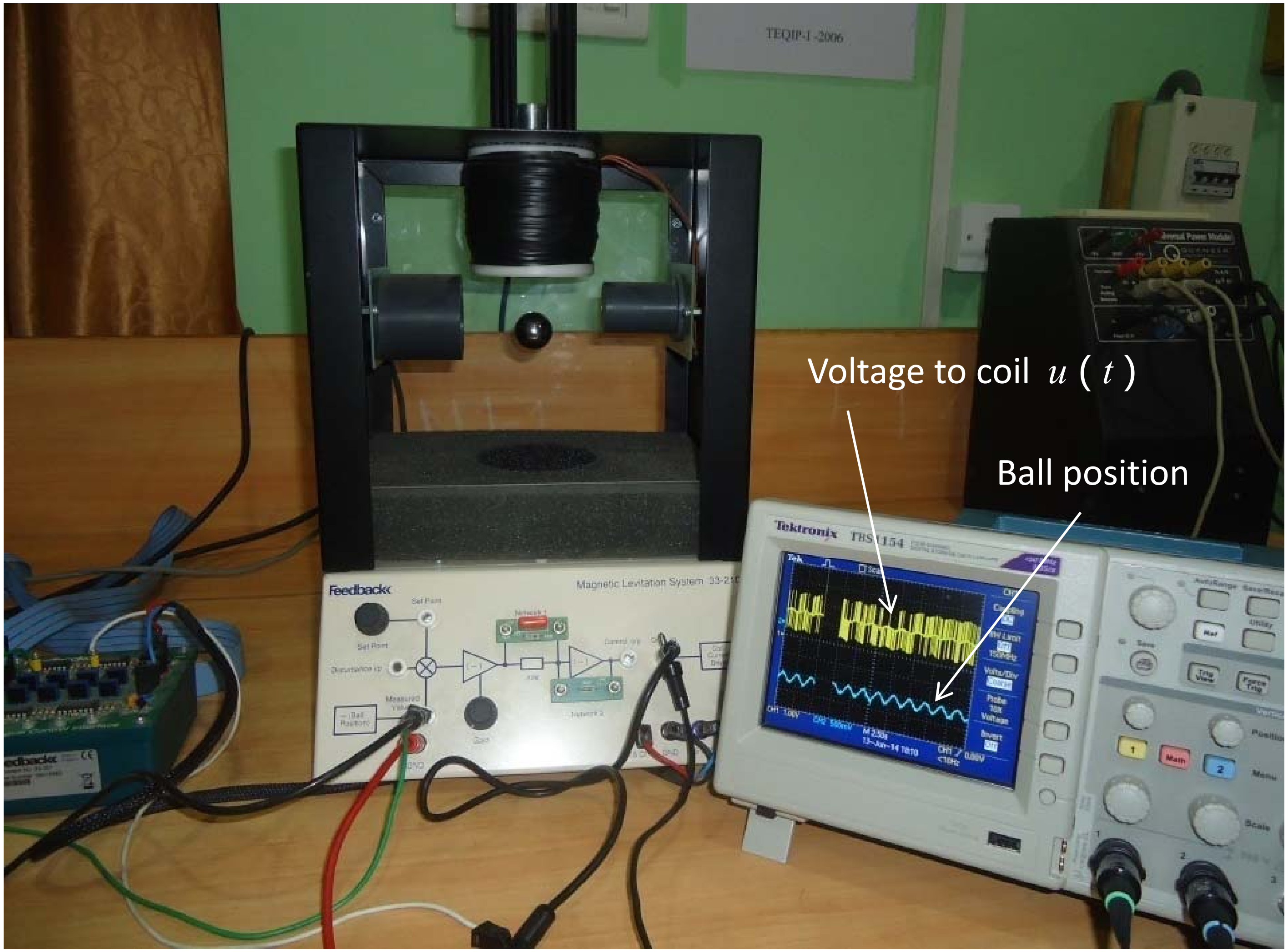
$$C_{FOPID}(s) = k_p + \frac{k_i}{s^{\alpha}} + k_d s^{\beta}$$

$$\alpha > 0 \qquad \beta > 0$$

$r(t)$: set point sinusoidal
$c(t)$: position of the levitated ball
$u(t)$: out-put of controller

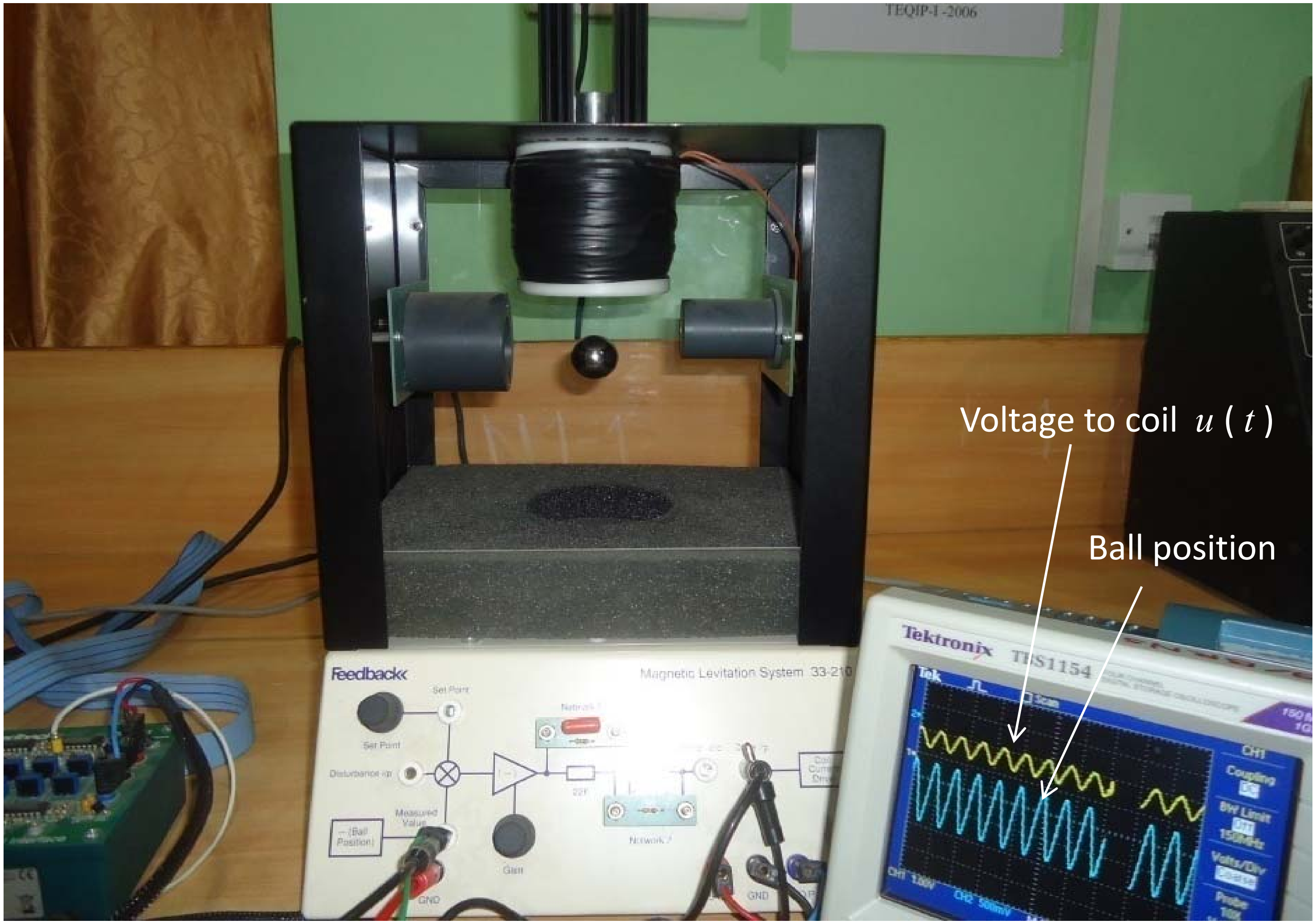
We note that Mag-Lev is inherent unstable unit

Control effort in case of PID control



Courtesy: BRNS funded joint project of VNIT Nagpur and BARC to develop digital fractional order controller for industrial applications.

Control effort in case of Fractional Order PID Controls



Courtesy: BRNS funded joint project of VNIT Nagpur and BARC to develop digital fractional order controller for industrial applications.

Why this control effort less? and its repercussion-a conjecture

To do the same job-that is to position the floating ball and slowly making its position follow sinusoidal command the output of the controller in case of first case is fluctuating severely.

The reason is that fractional derivatives and fractional integrals are having inherent memory .

This memory in the systems works to govern the ball position based on its previous experience-therefore these fractional differentiation and integration gives an ideal filtering action.

Whereas the classical differentiation is a point property-does not therefore has memory, and acts instantly with no previous experience. Thus in the first case the maneuvering signal is going very high instantaneously and in the next moment going very low again and again-lot of effort?

So we can see fractional calculus based system does the control action with a lesser effort than the conventional classical calculus based controllers, therefore are better efficient.

Can we say Fuel Efficient Controls?

Liouville's Postulation

In parallel to classical approach Joseph Liouville postulated exponential approach

$$\frac{\mathrm{d}^{\alpha} e^{ax}}{\mathrm{d}x^{\alpha}} = a^{\alpha} e^{ax}; \quad \alpha \in \mathbb{R}$$

Negative values of α represent integrations (anti-derivative) and we can even extend this to allow complex values of or α even to a continuous distribution of this order in some interval.

$$\frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}x^{\alpha}} \qquad \frac{\mathrm{d}^{-\beta} f(x)}{\mathrm{d}x^{-\beta}} \qquad \frac{\mathrm{d}^{\alpha+i\beta} f(x)}{\mathrm{d}x^{\alpha+i\beta}} \qquad \sum_{n} a_n \mathrm{D}_x^{\alpha_n} f(x) \qquad \int_a^b \left(K(\alpha) \frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}x^{\alpha}} \right) \mathrm{d}\alpha$$

In a way this generalizes notion of integration and differentiation to arbitrary order!

Liouvelli's class of functions and approach

Any function expressible as a sum of exponential functions can then be differentiated in the same way.

$$\frac{\mathrm{d}^{\alpha}\cos(x)}{\mathrm{d}x^{\alpha}} = \frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\left(\frac{e^{ix}+e^{-ix}}{2}\right) = \frac{(i)^{\alpha}e^{ix}+(-i)^{\alpha}e^{-ix}}{2}$$

$$= \frac{\left(e^{i\alpha\frac{\pi}{2}}\right)\left(e^{ix}\right)+\left(e^{-i\alpha\frac{\pi}{2}}\right)\left(e^{-ix}\right)}{2} = \frac{e^{i\left(x+\alpha\frac{\pi}{2}\right)}+e^{-i\left(x+\alpha\frac{\pi}{2}\right)}}{2}$$

$$= \cos\left(x+\alpha\tfrac{\pi}{2}\right)$$

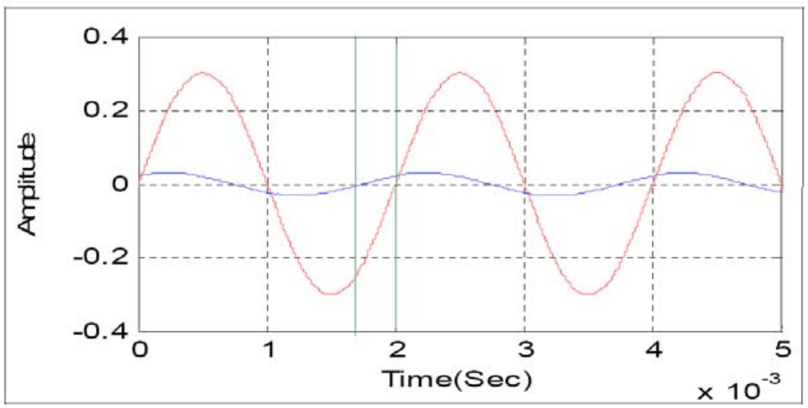
Since $(\pm i)^{\alpha} = \left(e^{\pm i\pi/2}\right)^{\alpha} = e^{\pm i\alpha\frac{\pi}{2}}$ we have the nice result $\dfrac{\mathrm{d}^{\alpha}\cos(x)}{\mathrm{d}x^{\alpha}} = \cos\left(x+\alpha\tfrac{\pi}{2}\right)$

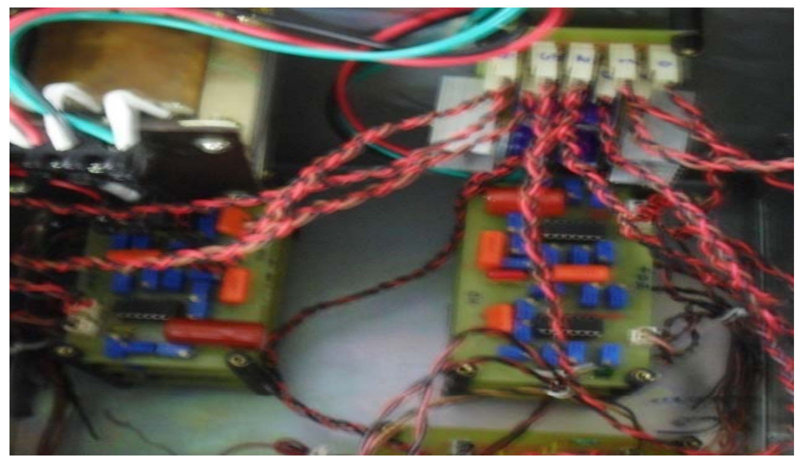
Is it so simple ?

Fractional order differentiator and integrator circuit realized

Liouville's exponential approach allows us to simply write $$\frac{d^\alpha \cos(x)}{dx^\alpha} = \cos\left(x + \alpha\frac{\pi}{2}\right)$$

Thus the generalized differential operator simply shifts the phase of the cosine function and likewise the sine function by $\alpha \times (90)^0$, that is in proportion to the order of the differentiation. For differentiation the process advances the phase, needless to say the integration makes the phase lagged. We test by giving a sinusoid at the input of the circuit and measure the output and record the phase lag or lead, depending on the fractional order.





Output of fractional order differentiator circuit for ½ order

Courtesy: BRNS funded joint project of VNIT Nagpur and BARC to develop analog fractional order PID.

Generalized Factorial to Gamma Function

Factorial is for positive Integer number is

$$n! = \begin{cases} 1 & n = 0 \\ (n-1)!(n) & n > 0 \end{cases}$$

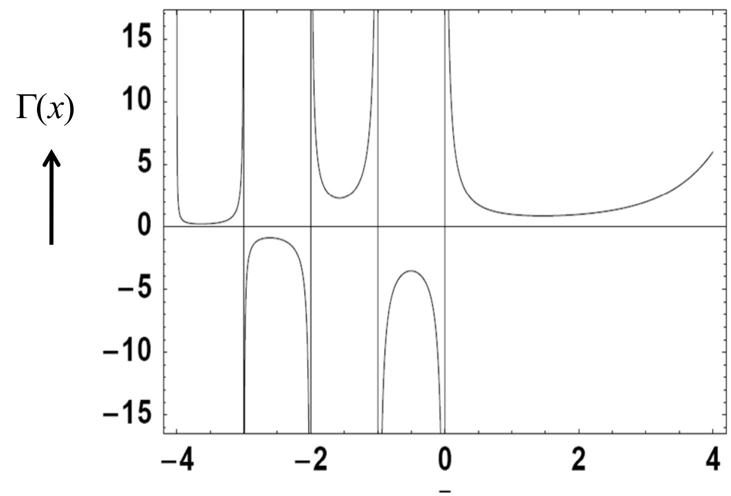
Euler's Gamma function

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \qquad \mathrm{Re}[\alpha] > 0$$

$$\Gamma(\alpha + 1) = \alpha \left(\Gamma(\alpha)\right)$$

This gives analytic continuity to negative axis

We write thus $\alpha! = \Gamma(\alpha + 1)$



Plot of Gamma-Function

Some values are following

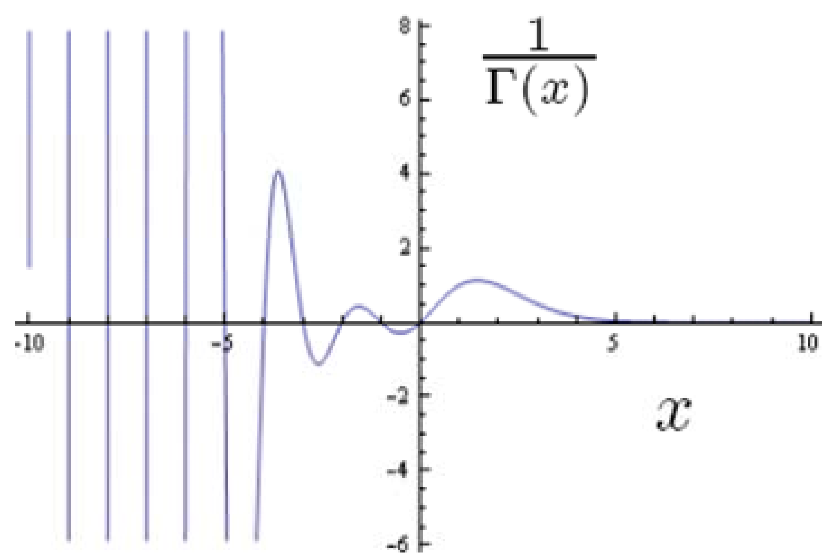
$$\left(\tfrac{3}{2}\right)! = \Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{4}\sqrt{\pi}$$

$$\left(-\tfrac{5}{2}\right)! = \Gamma\left(-\tfrac{3}{2}\right) = \tfrac{4}{3}\sqrt{\pi}$$

$$\left(\tfrac{1}{2}\right)! = \Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\sqrt{\pi}$$

$$\left(-\tfrac{1}{2}\right)! = \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\left(-\tfrac{3}{2}\right)! = \Gamma\left(-\tfrac{1}{2}\right) = -2\sqrt{\pi}$$



Plot of reciprocal of Gamma Function

Euler formula repeated differentiation

This is exactly what we would expect based on a straightforward interpolation of the derivatives of a power of x . Recalling that the first few (whole) derivatives of x^m are

$$\frac{\mathrm{d}x^m}{\mathrm{d}x} = m x^{m-1}; \quad \frac{\mathrm{d}^2 x^m}{\mathrm{d}x^2} = m(m-1)x^{m-2}; \quad \frac{\mathrm{d}^3 x^m}{\mathrm{d}x^3} = m(m-1)(m-2)x^{m-3}$$

Thus we expect to find that the general form of the n-th derivative of x^m is

$$\frac{\mathrm{d}^n x^m}{\mathrm{d}x^n} = \frac{m!}{(m-n)!} x^{m-n}$$

Replacing the integer n with the general value α , and using the gamma function to express the factorial, this suggests that the a fractional derivative of x^m is simply

$$\frac{\mathrm{d}^\alpha x^m}{\mathrm{d}x^\alpha} = \frac{m!}{(m-\alpha)!} x^{m-\alpha} = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha}$$

Euler's formula

$$\frac{\mathrm{d}^{1/2} x}{\mathrm{d}x^{1/2}} = \frac{1!}{\Gamma\left(1-\frac{1}{2}+1\right)} x^{1-\frac{1}{2}} = \frac{1}{\Gamma\left(\frac{3}{2}\right)} = 2\sqrt{\frac{x}{\pi}}$$

$$\frac{\mathrm{d}^{1/2}[1]}{\mathrm{d}x^{1/2}} = \frac{0!}{\Gamma\left(0-\frac{1}{2}+1\right)} x^{0-\frac{1}{2}} = \frac{x^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{\pi x}}$$

Half derivative of constant as non-zero !

Use of Euler's formula

Now, since analytic functions can be expanded into power series $f(x)=\sum_k a_k x^k$ we can use Euler formula, applying it term by term to determine the fractional derivatives of all such functions. Furthermore, applying this formula with negative values of α gives a plausible expression for the fractional- integral of a power of x. For example, to find the whole integral of x^3 we set $m=3$ and then compute via Euler formula as

$$\frac{d^{\alpha} x^{m}}{dx^{\alpha}} = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha}$$

$$\frac{d^{-1} x^{3}}{dx^{-1}} = \frac{3!}{\Gamma\left(3-(-1)+1\right)} x^{3-(-1)} = \frac{3!}{\Gamma(5)} x^{4} = \frac{6}{24} x^{4} = \frac{1}{4} x^{4}$$

Note that the above integration is valid only if the initial point be zero, else initial value is subtracted. The unification of these two operations makes it even less surprising that

generalized differentiation is non-local, just as is integration-has memory history hereditary

$$\frac{d^{\pm\alpha} x^{m}}{dx^{\pm\alpha}} = \frac{\Gamma(m+1)}{\Gamma(m\mp\alpha+1)} x^{m\mp\alpha}$$

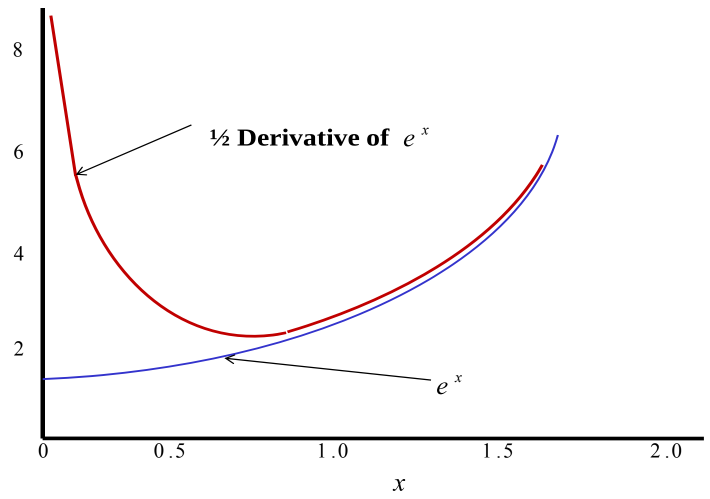
Fractional Derivative of Exponential Function is not Exponential !!

We previously proposed that the general α- th derivative of e^{ax} is simply , $a^{\alpha} e^{ax}$ and yet if we expand the exponential function e^{x} into a power series

$$e^{ax} = 1 + \frac{a}{1!}x + \frac{a^2}{2!}x^2 + \frac{a^3}{3!}x^3 + \ldots$$

and apply Euler formula to determine the half-derivative, term by term, we get (not at all what we postulated earlier!) Liouville's postulate , that is following

$$\frac{d^{1/2}(e^x)}{dx^{1/2}} = \frac{1}{\sqrt{\pi x}}\left(1 + 2x + \frac{4}{3}x^2 + \frac{8}{15}x^3 + \frac{16}{105}x^4 + \ldots\right)$$



Here we have $D^{1/2} e^x \neq e^x$

Was Liouville wrong or is there a contradiction?

Most fundamental approach

To get a clearer idea of the ambiguity in the concept of a generalized derivative, it's useful to examine a few other approaches, and compare them with the exponential approach of Liouville.

The most fundamental approach may be to begin with the basic definition of the whole derivative of a function

$$\frac{d\; f(x)}{dx} = \lim_{\varepsilon \downarrow 0} \frac{f(x) - f(x-\varepsilon)}{\varepsilon}$$

Repeating n -times of this operation leads to a binomial series of following type

$$\frac{d^n f(x)}{dx^n} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(x - j\varepsilon)$$

Note that ${}^nC_j = 0$ $\quad j > n$ thus summation above ends at n

Generalizing the binomial coefficients to real numbers we get a formula

$$\frac{d^\alpha f(x)}{dx^\alpha} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^\alpha} \sum_{j=0}^{\left[\frac{x-x_0}{\varepsilon}\right]} (-1)^j \frac{\Gamma(\alpha+1)}{j!\,\Gamma(\alpha+1-j)} f(x - j\varepsilon)$$

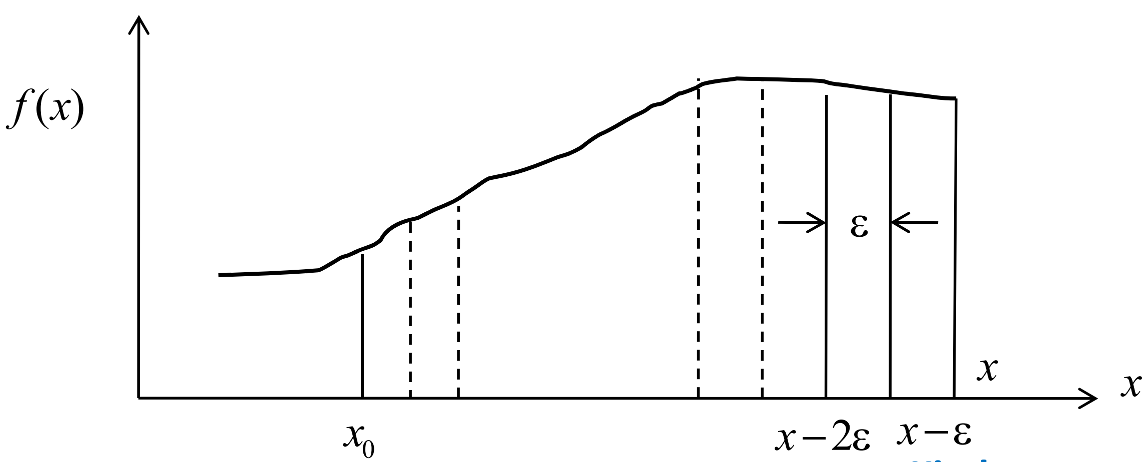
Non-locality of Generalized differ-integration

$$\frac{d^{\alpha} f(x)}{dx^{\alpha}} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{\alpha}} \sum_{j=0}^{\left\lfloor \frac{x-x_0}{\varepsilon} \right\rfloor} (-1)^j \frac{\Gamma(\alpha+1)}{j!\,\Gamma(\alpha+1-j)} f(x-j\varepsilon)$$

Thus, the generalized derivative is a non-local operation, just as is integration.

This can be seen from the factor $f(x-j\varepsilon)$ in the summation formula, showing that as j ranges from zero to $(x-x_0)/\varepsilon$ the argument of f ranges from x down to zero (or the start point of origination of function).

The fractional differ-integration is having memory



Fractional Derivatives Require Lower & Upper Limit like Integration !

Consider the two anti-differentiations (integrations) shown below

$$\int_{x_0}^{x} u^3 \,\mathrm{d}u = \frac{x^4}{4} - \frac{x_0^4}{4} \qquad\qquad \int_{x_0}^{x} e^u \,\mathrm{d}u = e^x - e^{x_0}$$

The first integral shows that when we say x^3 is the derivative of $x^4/4$ we are implicitly assuming $x_0 = 0$, which is consistent with our derivation of equation .

$$\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^\alpha} \sum_{j=0}^{\lfloor x/\varepsilon \rfloor} (-1)^j \frac{\Gamma(\alpha+1)}{j!\,\Gamma(\alpha+1-j)} f(x - j\varepsilon) \qquad x_0 = 0$$

However, the second integral shows that, by saying e^x is the derivative of e^x , we are implicitly assuming $x_0 = -\infty$:

$$\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^\alpha} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(\alpha+1)}{j!\,\Gamma(\alpha+1-j)} f(x - j\varepsilon)$$

We represent the operation with lower and higher terminals as:

$${}_a\mathrm{D}_x^\alpha f(x) \qquad {}_0\mathrm{D}_x^\alpha f(x) \qquad {}_{-\infty}\mathrm{D}_x^\alpha f(x) \qquad {}_x\mathrm{D}_b^\alpha f(x) \qquad {}_x\mathrm{D}_\infty^\alpha f(x)$$

For any arbitrary $\alpha \in \mathbb{R}$

Repeated integration approach

$$\frac{d^{-2}f(x)}{dx^{-2}} = \int_0^x \int_0^{u_1} f(u_1)du_2 du_1 = \int_0^x (x-u)f(u)du \; ; \quad \frac{d^{-3}f(x)}{dx^{-3}} = \int_0^x \int_0^{u_1} \int_0^{u_2} f(u_1)du_3 du_2 du_1 = \frac{1}{2!}\int_0^x (x-u)^2 f(u)du$$

$$\frac{d^{-n}f(x)}{dx^{-n}} = \underbrace{\int_0^x \int_0^{u_1} \int_0^{u_2} \ldots\ldots\ldots\ldots \int_0^{u_{n-1}}}_{n} f(u_1)du_n du_{n-1}\ldots\ldots du_1 = \frac{1}{(n-1)!}\int_0^x (x-u)^{n-1} f(u)du$$

$$= \frac{1}{\Gamma(n)}\int_0^x (x-u)^{n-1} f(u)du$$

Thus we have Cauchy's expression for repeated integrals, which is

$$\underbrace{\int_0^x \int_0^{u_1} \int_0^{u_2} \ldots\ldots\ldots\ldots \int_0^{u_{n-1}}}_{n} f(u_1)du_n du_{n-1}\ldots\ldots du_1 = \frac{1}{(n-1)!}\int_0^x (x-u)^{n-1} f(u)du$$

which we can express using the gamma function instead of factorials, for fractional order α as

$$\frac{d^{-\alpha}f(x)}{dx^{-\alpha}} = {}_0D_x^{-\alpha} f(x) \frac{1}{\Gamma(\alpha)}\int_0^x (x-u)^{\alpha-1} f(u)du$$

This is Riemann fractional integral formula.

Illustration of semi-integration by Riemann formula

Now let us do semi integration of $\sqrt{x}$, take then $\alpha=1/2$ and apply Riemann formula

$$
{}_0D_x^{-1/2}\sqrt{x} = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \frac{\sqrt{u}}{(x-u)^{\left(-\frac{1}{2}\right)+1}}du
$$

$$
= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \frac{u\,du}{\sqrt{u}\left(\sqrt{x-u}\right)}
$$

$$
= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \frac{u}{\sqrt{xu-u^2}}du
$$

$$
= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \frac{u\,du}{\sqrt{\frac{x^2}{4}-u^2-\frac{x^2}{4}+2x\left(\frac{u}{2}\right)}}
$$

$$
= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \frac{u\,du}{\sqrt{\frac{x^2}{4}-\left(u-\frac{x}{2}\right)^2}}
$$

Put $u = \frac{x+x\sin\theta}{2}$ $\quad du = \left(\frac{x}{2}\right)(\cos\theta)d\theta$

$u=0;\ \theta=-\pi/2$ $\quad u=x;\ \theta=+\pi/2$

$$
{}_0D_x^{-1/2}\left[\sqrt{x}\right] = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{-\pi/2}^{+\pi/2} d\theta\,\frac{\left(\frac{x}{2}\right)(\cos\theta)\left(\frac{x+x\sin\theta}{2}\right)}{\sqrt{\frac{x^2}{4}-\frac{x^2}{4}\sin^2\theta}}
$$

$$
= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{-\pi/2}^{+\pi/2} d\theta\,\frac{\left(\frac{x}{2}\right)(\cos\theta)\left(\frac{x+x\sin\theta}{2}\right)}{\left(\frac{x}{2}\right)\sqrt{1-\sin^2\theta}}
$$

$$
= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{-\pi/2}^{+\pi/2} d\theta\left(\frac{x+x\sin\theta}{2}\right) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[\frac{x\theta}{2}-\frac{x\cos\theta}{2}\right]_{\theta=-\pi/2}^{\theta=+\pi/2}
$$

$$
= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{\pi x}{2}\right)
$$

$$
= \frac{\sqrt{\pi}}{2}x
$$

Rigorous indeed

we get the same answer as from Euler formula $\frac{d^{-\alpha}x^m}{dx^{-\alpha}} = \frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)}x^{m+\alpha}$

$$
{}_0D_x^{-1/2}\left[x^{1/2}\right] = \frac{\Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+1\right)}x^{\frac{1}{2}+\frac{1}{2}} = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}x = \Gamma\left(\frac{3}{2}\right)x = \frac{\sqrt{\pi}}{2}x
$$

Riemann-Liouvelli and Caputo Fractional Derivative

Riemann fractional integral formula is $\frac{d^{-\alpha} f(x)}{dx^{-\alpha}} = {}_0D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} f(u)\,du$

There are two different ways in which this formula might be applied. For example, if we wish to find the (7/3)-rd ($\mu = 7/3$)derivative of a function (i.e. $d^{7/3} f(x)/dx^{7/3}$), we could begin by differentiating the function three whole times (taking nearest integer say m just greater than μ that is $m=3$), and then apply the above formula with $\alpha = (m-\mu) = \left(3-\left(\frac{7}{3}\right)\right) = \frac{2}{3}$ to "deduct" two thirds i.e. $\alpha = 2/3$ of a anti-differentiation

$$ {}_a^C\mathrm{D}_x^{7/3}\left[f(x)\right] = {}_a\mathrm{D}_x^{-2/3}\left[f^{(3)}(x)\right] \qquad \alpha = 3-\left(\tfrac{7}{3}\right) $$

$$ {}_a^C\mathrm{D}_x^{\mu}\left[f(x)\right] = {}_a\mathrm{D}_x^{-\alpha}\left[f^{(m)}(x)\right] \qquad \alpha = m-\mu $$

Caputo 1967

By Caputo we get fractional derivative of constant as zero-but f needs be differentiable

Alternatively we could begin by applying the above formula with $\alpha = 2/3$ and then differentiate the resulting function three whole times ($m=3$).

$$ {}_a^{RL}\mathrm{D}_x^{7/3}\left[f(x)\right] = \frac{d^3}{dx^3}\left[{}_a\mathrm{D}_x^{-2/3} f(x)\right] \qquad \alpha = 3-\left(\tfrac{7}{3}\right) $$

$$ {}_a^{RL}\mathrm{D}_x^{\mu}\left[f(x)\right] = \frac{d^m}{dx^m}\left[{}_a\mathrm{D}_x^{-\alpha} f(x)\right] \qquad \alpha = m-\mu $$

Riemann-Liouville 1872

Integral Representation of RL and Caputo Derivative

Caputo Derivative

$$f^{(\alpha)}(x) = \mathrm{D}^{-(1-\alpha)}\left(\mathrm{D}^{(1)} f(x)\right) \qquad 0 < \alpha < 1 \qquad m = 1$$

$${}_{a}^{\mathrm{C}}\mathrm{D}_{x}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_{a}^{x} \left(x-y\right)^{-\alpha} f^{(1)}(y)\mathrm{d}y\,, \quad 0 < \alpha < 1$$

Riemann- Liouville fractional derivative

$$f^{(\alpha)}(x) = \mathrm{D}^{(1)}\left(\mathrm{D}^{-(1-\alpha)} f(x)\right) \qquad 0 < \alpha < 1 \qquad m = 1$$

$${}_{a}\mathrm{D}_{x}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_{a}^{x} \left(x-y\right)^{-\alpha} f(y)\mathrm{d}y\,, \quad 0 < \alpha < 1$$

Riemann-Lioville (RL) & Caputo pros and cons !!

Although these two definitions give the same result in many circumstances especially when the start point of the process is at $-\infty$ or value of function at start point is zero.

They are not entirely equivalent, because (for example) the half-derivative of a constant is zero by the Caputo, whereas the RL gives for the half-derivative of a constant the result given previously as Euler formula.

The function requires to be differentiable for Caputo fractional derivative while the function need not be differentiable for RL fractional derivative

The relation between RL and Caputo is

$$ {}_{a}\mathrm{D}_{x}^{\alpha}\left[f(x)\right] = {}_{a}^{C}\mathrm{D}_{x}^{\alpha}\left[f(x)\right] + f(a)\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} \qquad 0<\alpha<1 $$

$$ {}_{a}^{C}\mathrm{D}_{x}^{\alpha}\left[f(x)\right] = {}_{a}\mathrm{D}_{x}^{\alpha}\left[f(x)-f(a)\right] $$

Fractional Derivatives of exponential function with lower terminal

We have for exponential function derivative with start point $x_0 = a$ as

$$ {}_a\mathrm{D}_x^{\alpha}\left[e^{C-cx}\right] = \frac{e^{C-cx}}{(x-a)^{\alpha}}\left(\gamma^*\left(-\alpha,-c(x-a)\right)\right) $$

This is obtained via RL formula

$$ {}_0\mathrm{D}_x^{1/2}\left[e^{\pm x}\right] = \frac{e^{\pm x}}{x^{1/2}}\gamma^*\left(-\tfrac{1}{2},\pm x\right) \qquad \gamma^*(\alpha,x) \overset{\mathrm{def}}{=} \frac{x^{-\alpha}}{\Gamma(\alpha)}\int_0^x e^{-t}t^{\alpha-1}\mathrm{d}t $$

γ^* is Tricomi's incomplete Gamma function

For start point as $x_0 = -\infty$

$$ {}_{-\infty}\mathrm{D}_x^{\alpha}\left[e^{cx}\right] = \frac{\mathrm{d}^{\alpha}\left[e^{cx}\right]}{\left[\mathrm{d}(x+\infty)\right]^{\alpha}} = \frac{e^{cx}}{(x+\infty)^{\alpha}}\gamma^*\left(-\alpha,c(x+\infty)\right) $$

$$ = e^{cx}\lim_{y\to\infty}\left(\frac{\gamma^*(-\alpha,cy)}{y^{\alpha}}\right) \qquad x+\infty = y $$

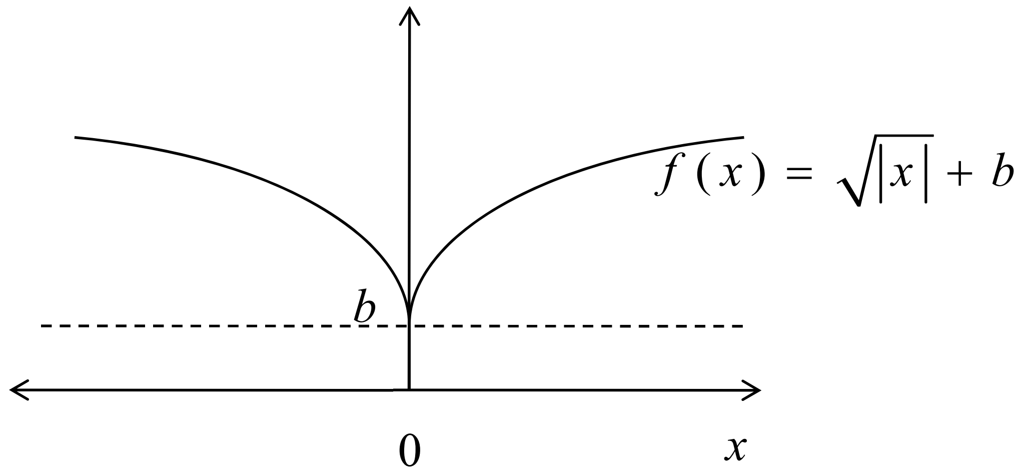
$$ = e^{cx}\lim_{y\to\infty}\left(c^{\alpha}\frac{\gamma^*(-\alpha,cy)}{(cy)^{\alpha}}\right) = c^{\alpha}e^{cx}\lim_{z\to\infty}\left(\frac{\gamma^*(-\alpha,z)}{z^{\alpha}}\right) \qquad cy = z $$

$$ = c^{\alpha}e^{cx}\left(1-\frac{e^{-z}}{z^{\alpha+1}\Gamma(-\alpha)}\left(1-\tfrac{\alpha+1}{z}+O(z^{-2})\right)\right) $$

$$ = c^{\alpha}e^{cx} $$

We have got Liouvelli's postulate

Fractional derivative of non-differentiable function



$$ {}_0D_x^{1/2}[f(x)] = {}_0D_x^{1/2}\left[\sqrt{|x|} + b\right] $$

$$ = \frac{\Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(\frac{1}{2}+1-\frac{1}{2}\right)}|x|^{1/2-(1/2)} + b\,\frac{|x|^{-1/2}}{\Gamma\left(1-\frac{1}{2}\right)} $$

$$ = \Gamma\left(\tfrac{3}{2}\right) + \frac{b}{\Gamma\left(\frac{1}{2}\right)}|x|^{-1/2} $$

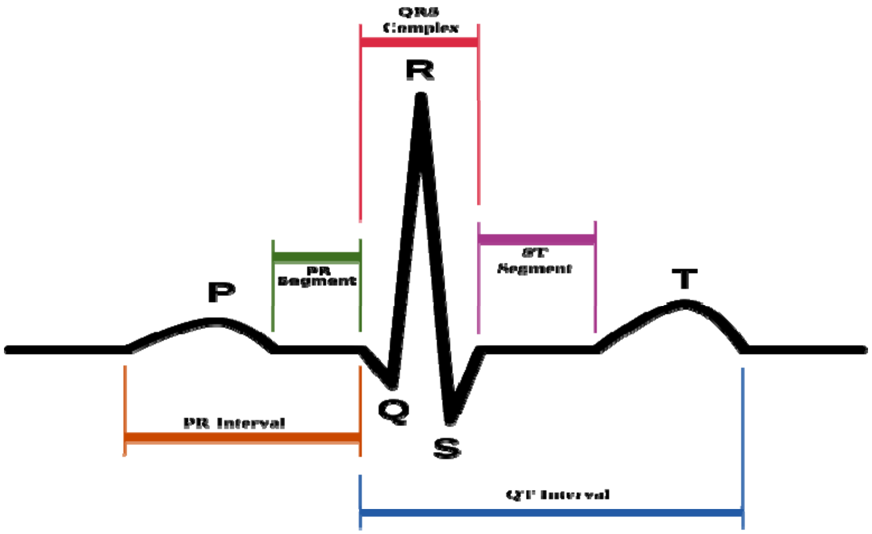
If we offset the constant then we get

$$ {}_0D_x^{1/2}[f(x) - f(0)] = {}_0D_x^{1/2}\left[\left(\sqrt{|x|} + b\right) - b\right] $$

$$ = \Gamma\left(\tfrac{3}{2}\right) = \left(\tfrac{1}{2}\right)! $$

we get fractional derivative value at the non-differentiable points

Characterizing non-differentiable points of ECG graph



We have done studies on two normal ECG graphs and ten ECG graphs of LVH patients, by finding fractional derivative (of half order), phase transition values at non-differentiable points, leads.

The values of P.T. of normal ECG leads for different non-differentiable points are low but it increases abruptly f or LVH patients

This type of study is not reported elsewhere. This method is a new method we are reporting for the first time-could be an aid for differential diagnostics in medical science.

Courtesy :Dept. Appl. Mathematics Calcutta Univ. BRNS Funded Project: Characterization of unreachable (Holderian) functions via Local Fractional Derivative and Deviation of function

RL Derivative to get L'Hopital's answer

In 1695, 30 Sept. L'Hospital asked Leibniz what if we put $n = ½$ in $D^n[x]$ is date of birth of FC

Leibniz replied an apparent paradox may lead to useful consequences

To illustrate the use of these RL-Caputo definitions we will determine the half-derivative of x as L'Hopital requested. Using the RL formula, we first apply half of an integration to this function using equation Riemann integration formula with $\alpha = 1/2$, giving

$$\frac{d^{-1/2}x}{dx^{-1/2}} = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x (x-u)^{-1/2}(u)du \qquad \text{put} \qquad x-u=z, \quad du=-dz$$

$$= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_x^0 (-dz)\frac{(x-z)}{\sqrt{z}} = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[\int_x^0 z^{1/2}dz - \int_x^0 xz^{-1/2}dz\right]$$

$$= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[-\tfrac{2}{3}x^{3/2} + 2x^{3/2}\right] = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{4x^{3/2}}{3}$$

$$= \frac{4}{3\sqrt{\pi}}x^{3/2}$$

Then we apply one whole differentiation to give the net result of a half-derivative

$$\frac{d^{1/2}x}{dx^{1/2}} = \frac{d}{dx}\left(\frac{4x^{3/2}}{3\sqrt{\pi}}\right) = 2\sqrt{\frac{x}{\pi}}$$

So ${}_0D^{½}[x] = 2\sqrt{\frac{x}{\pi}}$ via RL formula

Caputo Derivative to get L'Hopital's answer

In this case the Caputo gives the same result. Choose $m = 1$ then differentiate the function $f(x) = x$ once to have $f^{(1)}(x) = 1$, then do the semi-integration of this that is.

$$ {}_0\mathrm{D}_x^{-1/2}[1] = \Gamma(1)x^{1/2} / \Gamma\left(\tfrac{3}{2}\right) = 2\sqrt{\tfrac{x}{\pi}} $$

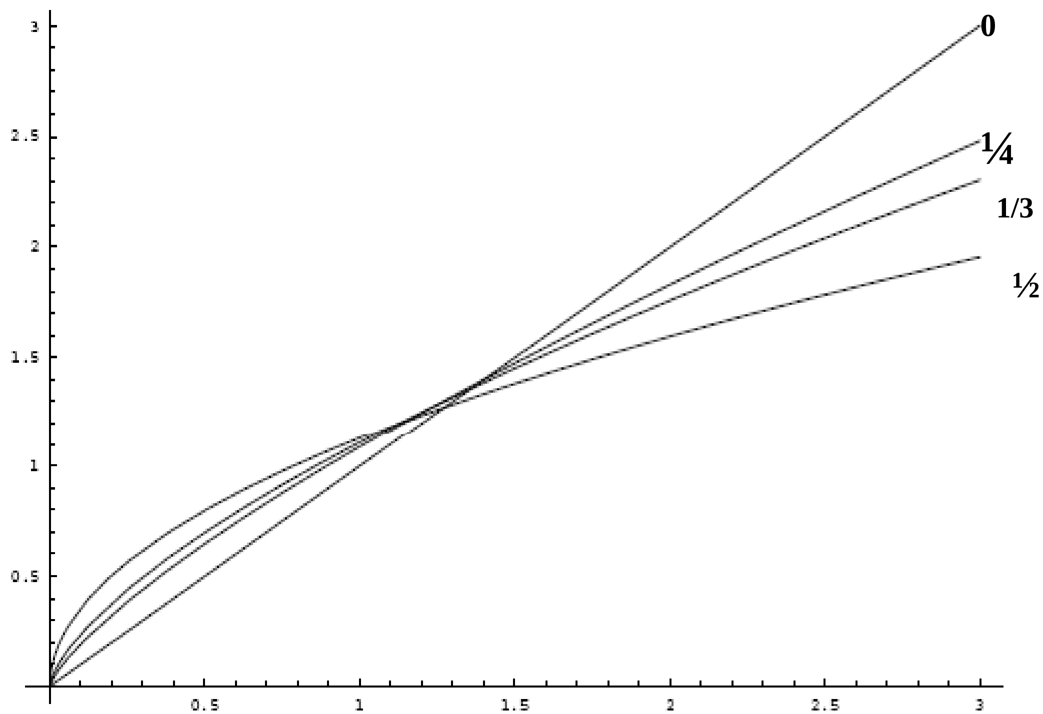
$$ {}_0^C\mathrm{D}_x^{1/2}[x] = {}_0\mathrm{D}_x^{-1/2}\left[\frac{\mathrm{d}(x)}{\mathrm{d}x}\right] $$

$$ = {}_0\mathrm{D}_x^{-1/2}[x^0] = \frac{\Gamma(1)}{\Gamma\left(\tfrac{3}{2}\right)} x^{0+\left(\frac{1}{2}\right)} = 2\sqrt{\frac{x}{\pi}} $$

So ${}_0^C\mathrm{D}^{½}[x] = 2\sqrt{\tfrac{x}{\pi}}$ via Caputo formula

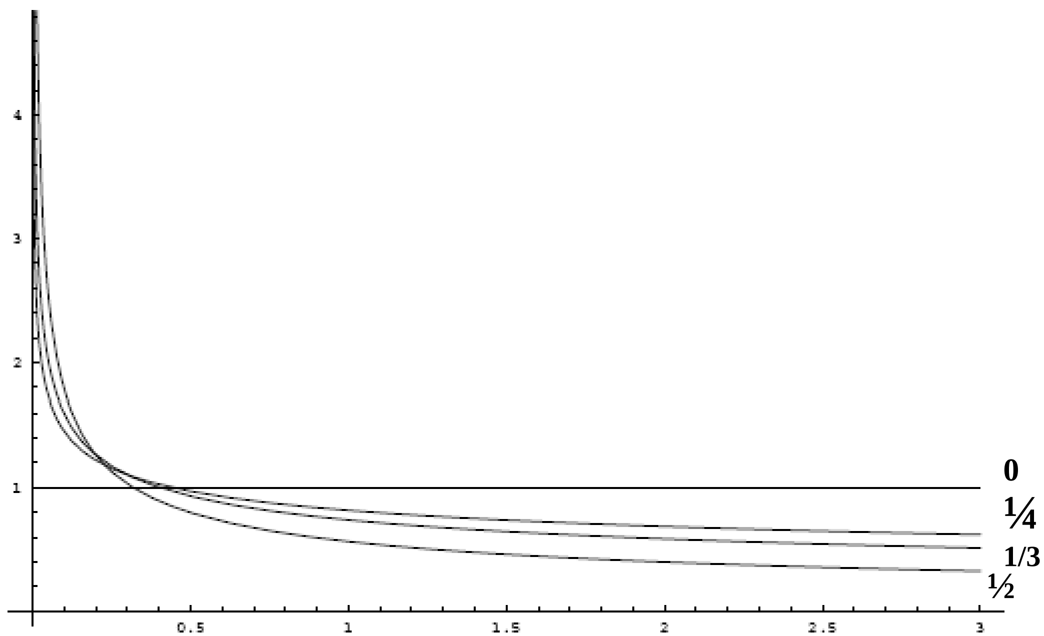
RL and Caputo gave same answer here since value of function at start point of fractional differ-integration in this case starts at zero is zero, i.e. at $x = 0,\ f(x) = 0$

Fractional derivatives of $f(x) = x$ for orders 0, ¼ , ….. , ½



Observe how derivatives approach $f(x) = 1$, but each derivative passes through origin

Fractional derivatives of $f(x) = 1$ for orders 0, ¼ , ….. , ½



Observe that the derivatives converge to $f(x) = 0$ except for an asymptote at $x = 0$

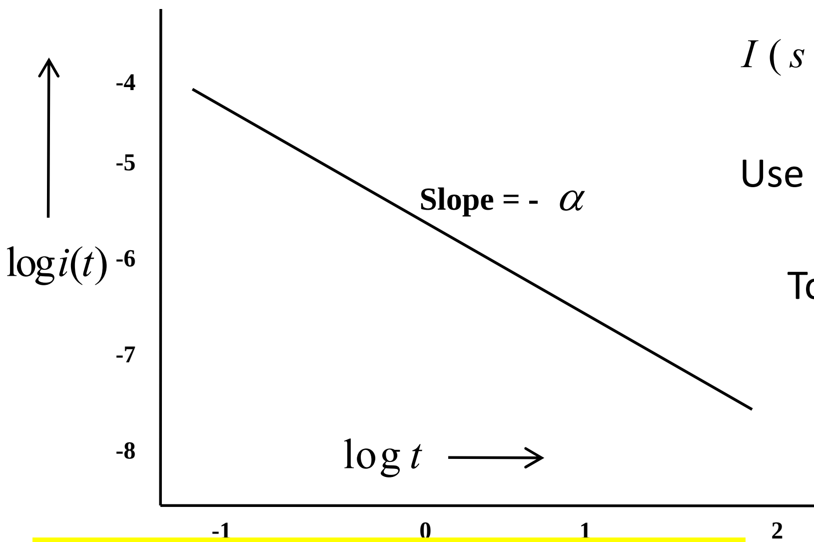
Expressing fractional impedance via Curie relaxation law

As in di-electric relaxation, the relaxation of current to an impressed constant voltage stress U_0 i.e. a step voltage applied at $t = 0$, to an initial uncharged system the current is by following law

$$i(t) = \frac{U_0}{h_\alpha t^\alpha} \qquad t > 0 \qquad 0 < \alpha < 1$$

This is Curie law 1889

We now get Transfer Function of a capacitor via Laplace Transform



Current vs time. At $t = 0$ a voltage of 100V is connected to a 0.47 uF capacitor with metalized paper dielectric gives $\alpha = 0.86$

$$I(s) = \mathcal{L}\{i(t)\} = \mathcal{L}\left\{\frac{U_0}{h_\alpha t^\alpha}\right\}$$

Use Laplace pair $\frac{n!}{s^{n+1}} \leftrightarrow t^n$ and $n! = \Gamma(1+n)$

To get $I(s) = U_0 \frac{\Gamma(1-\alpha)}{h_\alpha s^{-\alpha+1}} = \left(\frac{\Gamma(1-\alpha)}{h_\alpha s^{-\alpha}}\right)\left(\frac{U_0}{s}\right)$

Note that the voltage excitation is a constant step input at time zero

$$u(t) = \begin{cases} U_0 & t > 0 \\ 0 & t \le 0 \end{cases}$$

Laplace is $U(s) = \frac{U_0}{s}$

For unit step we have

$$H(s) = \frac{I(s)}{U(s)} = \frac{\Gamma(1-\alpha)}{h_\alpha s^{-\alpha}} = \frac{\Gamma(1-\alpha)}{h_\alpha} s^\alpha = C_\alpha s^\alpha$$

$$C_\alpha = \frac{\Gamma(1-\alpha)}{h_\alpha} \qquad Z(s) = \frac{U(s)}{I(s)} = \frac{1}{C_\alpha s^\alpha}$$

$$Z(\omega) = \frac{1}{C_\alpha (i\omega)^\alpha}$$

# Indicates fractional derivative in current-voltage expression

We have just expressed impedance in Laplace domain for capacitor as

$$Z(s) = \frac{U(s)}{I(s)} = \frac{1}{C_\alpha s^\alpha} \qquad 0 < \alpha < 1$$

Fractional order impedance

Unit of fractional capacity C_α is $\mathrm{F/s^{1-\alpha}}$ in this new relation !

This gives current-voltage expression for capacitor as

$$Z(s) = \frac{U(s)}{I(s)} = \frac{1}{C_\alpha s^\alpha} \qquad 0 < \alpha < 1$$

$$I(s) = \left(C_\alpha s^\alpha\right) U(s) = C_\alpha \left(s^\alpha U(s)\right)$$

As we have for zero initial condition $s\left(F(s)\right) = s\left(\mathcal{L}\{f(t)\}\right) \leftrightarrow \frac{\mathrm{d}\, f(t)}{\mathrm{d}t}$

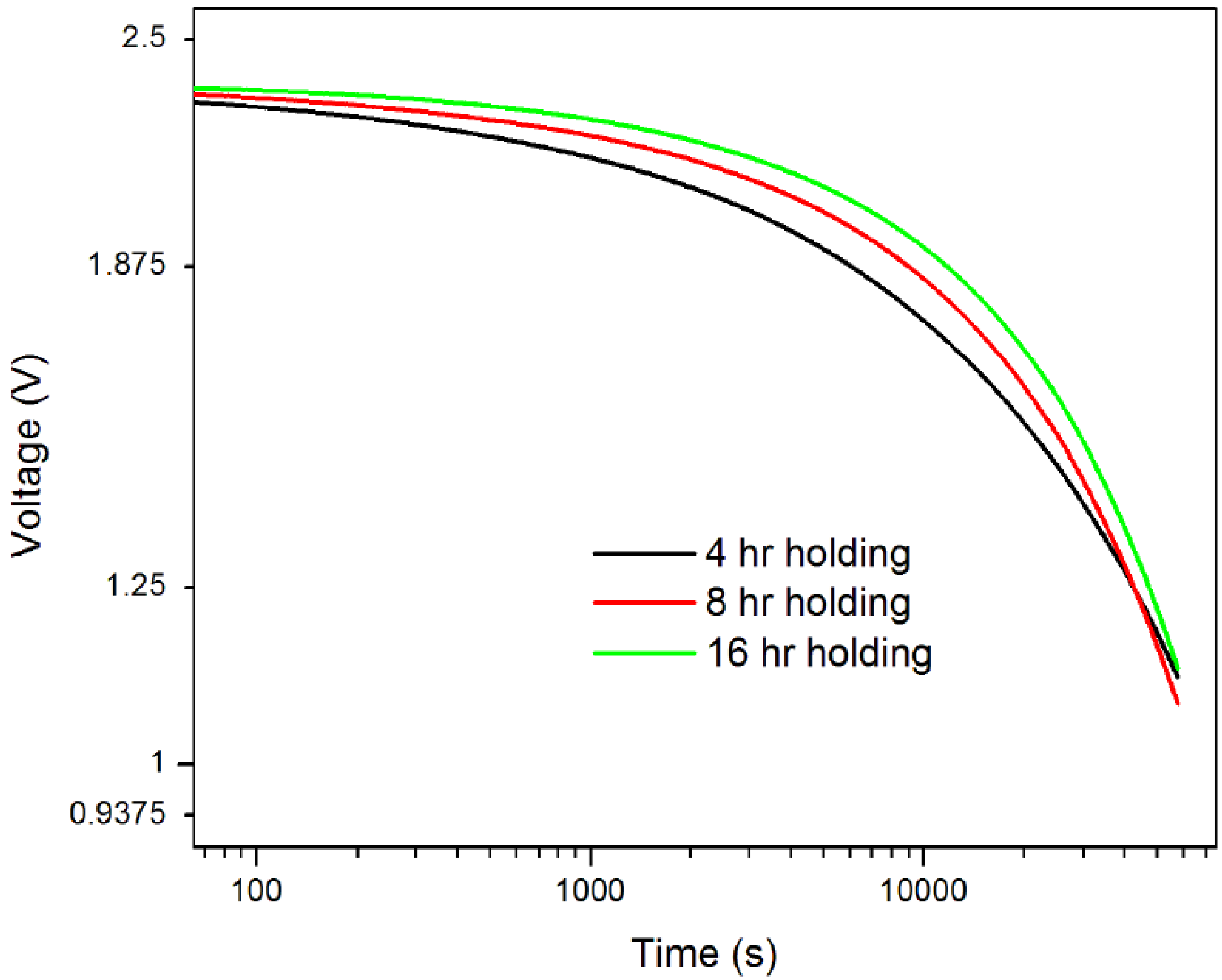
we generalize to fractional order derivative and write $s^\alpha\left(F(s)\right) = s^\alpha\left(\mathcal{L}\{f(t)\}\right) \leftrightarrow \frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}$

Our new-capacitor expression is $i(t) = C_\alpha \frac{\mathrm{d}^\alpha u(t)}{\mathrm{d}t^\alpha}$ $\qquad u(t) = \frac{1}{C_\alpha}\int_0^t i(\tau)\left(\mathrm{d}\tau\right)^\alpha$

Contrary to classical expression $i(t) = C\frac{\mathrm{d}\left(u(t)\right)}{\mathrm{d}t}$ $\qquad u(t) = \frac{1}{C}\int_0^t i(\tau)\left(\mathrm{d}\tau\right)$

The charging time is memorized

We charge a ultra-capacitor to maximum limit voltage and keep for time T on float charge, then it is kept under open-circuit. The self-discharge plot is depending on T



Courtesy: BRNS Funded joint Project CMET Thrissur-BARC Development of CAG Super-capacitors and application in electronics circuits

Self discharge with memory via fractional derivative for supercapacitor

A capacitor is charged from time $-T$ to t with a constant voltage U_0; the charging current is

$$i_c(t) = C_\alpha \left.\frac{\mathrm{d}^\alpha U_0}{\mathrm{d}t^\alpha}\right|_{-T}^{t} = \left(\frac{\Gamma(1-\alpha)}{h_\alpha}\right)\left.\frac{\mathrm{d}^\alpha U_0}{\mathrm{d}t^\alpha}\right|_{-T}^{t} = \frac{\Gamma(1-\alpha)}{h_\alpha}\left(\frac{U_0}{\Gamma(1-\alpha)}(t-(-T))^{-\alpha}\right)$$

$$= \frac{U_0}{h_\alpha (t+T)^\alpha} \qquad 0<\alpha<1 \qquad (t+T)>0$$

Using FD of constant i.e. ${}_a\mathrm{D}_x^\alpha C = C\,\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}$

At $t=0$ it is kept in open-circuit; there will be self discharge thus a discharge current will appear depending on decaying of the terminal voltage $u(t)$

$$i_d(t) = C_\alpha \left.\frac{\mathrm{d}^\alpha u(t)}{\mathrm{d}t^\alpha}\right|_{0}^{t} = \frac{\Gamma(1-\alpha)}{h_\alpha}\left.\frac{\mathrm{d}^\alpha u(t)}{\mathrm{d}t^\alpha}\right|_{0}^{t}$$

We combine the charge and self discharge together and write $i_c(t)+i_d(t)=0$

$$\frac{U_0}{h_\alpha (t+T)^\alpha} + \frac{\Gamma(1-\alpha)}{h_\alpha}\frac{\mathrm{d}^\alpha u(t)}{\mathrm{d}t^\alpha} = 0$$

Do fractional integration of order α for above expression, to write the following

$${}_0D_t^{-\alpha}\left[\frac{U_0}{h_\alpha (t+T)^\alpha}\right] + \frac{\Gamma(1-\alpha)}{h_\alpha}\left[u(t)-U_0\right] = 0$$

We apply the formula for fractional integration i.e. ${}_0D_t^{-\alpha}\left[f(t)\right] = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(x)\mathrm{d}x}{(t-x)^{1-\alpha}}$ to get

Contd...

Contd...

$$\frac{U_0}{h_\alpha \Gamma(\alpha)}\int_0^t \frac{1}{(T+x)^\alpha}\frac{\mathrm{d}x}{(t-x)^{1-\alpha}} + \left[\frac{\Gamma(1-\alpha)}{h_\alpha}u(t) - \frac{\Gamma(1-\alpha)}{h_\alpha}U_0\right] = 0$$

Rearranging the above, we write

$$u(t) = U_0 - \frac{U_0}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t \frac{\mathrm{d}x}{(T+x)^\alpha (t-x)^{1-\alpha}}$$

Put $T + x = \tau$ $\mathrm{d}x = \mathrm{d}\tau$ so for $x = 0$ $\tau = T$ and $x = t$ $\tau = T + t$ We have thus

$$u(t) = U_0 - \frac{U_0}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_T^{T+t} \frac{\mathrm{d}\tau}{\tau^\alpha (T+t-\tau)^{1-\alpha}} = U_0 - \frac{U_0}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_T^{T+t} F(\tau)\mathrm{d}\tau$$

Now we break $\int_T^{T+t} F(\tau)\mathrm{d}\tau$ as $\int_T^{T+t} F(\tau)\mathrm{d}\tau = \int_T^0 F(\tau)\mathrm{d}\tau + \int_0^{T+t} F(\tau)\mathrm{d}\tau$ and call the second term as $\mathcal{I}(t)$

We write in terms of convolution of two functions with substitution $T + t = \overline{t}$

$$\mathcal{I}(t) = \int_0^{T+t} F(\tau)\mathrm{d}\tau = \int_0^{T+t} \frac{\mathrm{d}\tau}{\tau^\alpha (T+t-\tau)^{1-\alpha}} = \int_0^{\overline{t}} \frac{\mathrm{d}\tau}{\tau^\alpha (\overline{t}-\tau)^{1-\alpha}} = \left(\frac{1}{t^\alpha}\right) * \left(\frac{1}{t^{1-\alpha}}\right)$$

Using Laplace pair $t^\alpha \leftrightarrow \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ we write $\mathcal{L}\left\{\mathcal{I}(t)\right\} = \mathcal{I}(s) = \mathcal{L}\left\{t^{-\alpha}\right\} \times \mathcal{L}\left\{t^{-(1-\alpha)}\right\}$ as

$$\mathcal{I}(s) = \frac{\Gamma(-\alpha+1)}{s^{-\alpha+1}} \times \frac{\Gamma\left(-(1-\alpha)+1\right)}{s^{[-(1-\alpha)+1]}} = \frac{\Gamma(1-\alpha)\Gamma(\alpha)}{s}$$

Extracting $\mathcal{I}(t)$ by inverse Laplace of obtained $\mathcal{I}(s)$ we get

$$\mathcal{I}(t) = \mathcal{L}^{-1}\left\{\mathcal{I}(s)\right\} = \mathcal{L}^{-1}\left\{\Gamma(1-\alpha)\Gamma(\alpha)\left(\tfrac{1}{s}\right)\right\} = \Gamma(1-\alpha)\Gamma(\alpha)$$

We used Laplace pair $1 \leftrightarrow \frac{1}{s}$

Contd...

Contd…

Using just derived expression i.e. $\int_0^{T+t} F(\tau)\mathrm{d}\tau = \int_0^{T+t} \frac{1}{\tau^\alpha (T+t-\tau)^{1-\alpha}} \mathrm{d}\tau = \Gamma(1-\alpha)\Gamma(\alpha)$ we write the expression for open circuit voltage $u(t)$ as follows

$$
\begin{aligned}
u(t) &= U_0 - \frac{U_0}{\Gamma(1-\alpha)\Gamma(\alpha)}\left[\int_0^{T+t} F(\tau)\mathrm{d}\tau + \int_T^0 F(\tau)\mathrm{d}\tau\right] \\
&= U_0 - \frac{U_0}{\Gamma(1-\alpha)\Gamma(\alpha)}\left[\Gamma(1-\alpha)\Gamma(\alpha)\right] - \frac{U_0}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_T^0 F(\tau)\mathrm{d}\tau \\
&= -\frac{U_0}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_T^0 F(\tau)\mathrm{d}\tau = \frac{U_0}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^T F(\tau)\mathrm{d}\tau \\
&= \frac{U_0}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^T \frac{\mathrm{d}\tau}{\tau^\alpha (T+t-\tau)^{1-\alpha}}
\end{aligned}
$$

Therefore

$$
u(t) = \frac{U_0}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^T \frac{\mathrm{d}\tau}{\tau^\alpha (T+t-\tau)^{1-\alpha}}
$$

is the voltage over open capacitor at self discharge. This function of time, depends on the total time T the capacitor has been on the voltage source.

In a way this capacitor is memorizing its charging history.

This explanation was possible only by usage of fractional derivative

Where can we find fractional capacitor?

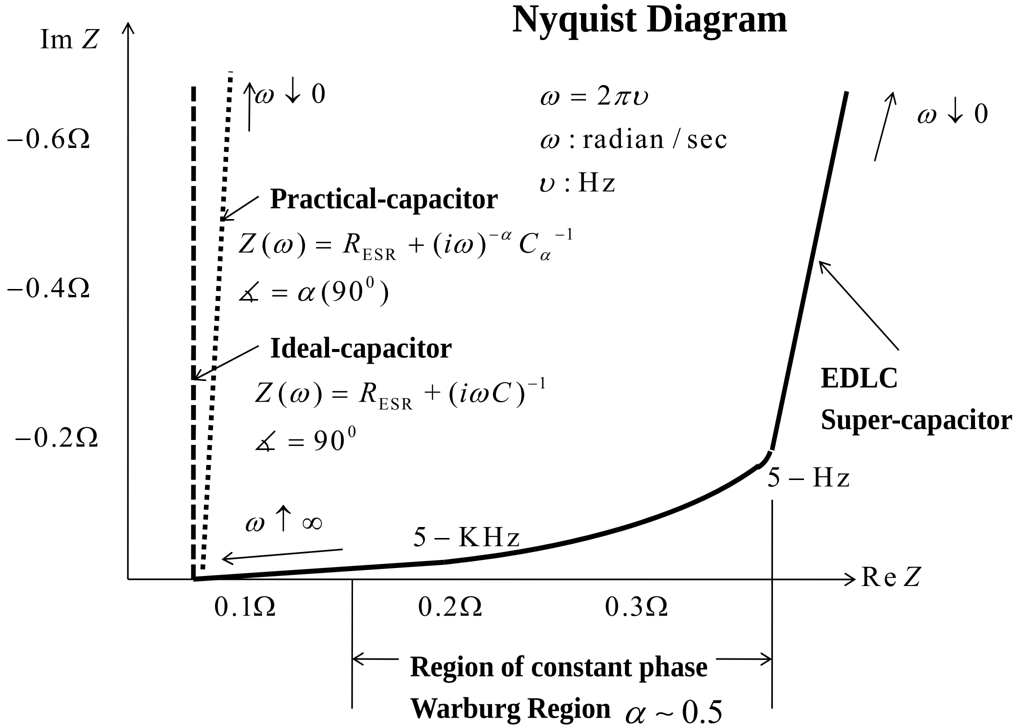
There are fractional order behaviors observed in

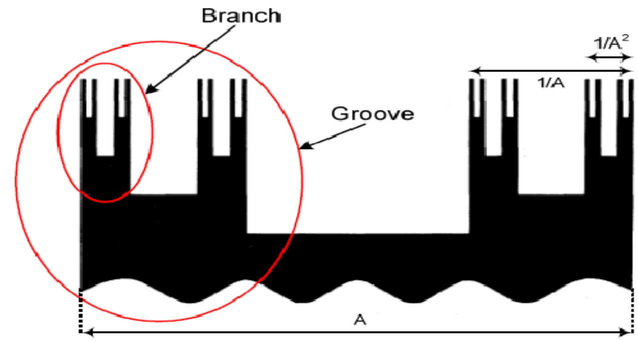
1. Insulation studies
2. Di-electric relaxations
3. Super-Capacitors made by Carbon-Aerogel, CNT, Graphine, Acivated Carbon, Conducting polymer electrodes
4. The studies on gelatin bio-polymers as electrolyte material
5. Lithium ion batteries
6. Studies on PMMA material.
7. Studies with fractal electrodes. etc

Fractional Order Capacitor

$$Z(s) = \frac{U(s)}{I(s)} = \frac{1}{C_\alpha s^\alpha}, \quad \alpha < 1$$

Impedance Spectroscopy





Self similar repeated structure of electrode-rough porous electrode

Fractal nature non-integer dimension

$2 < d_f < 3; \quad \alpha = (d_f - 1)^{-1}$

The perfect smooth surface $d_f = 2$, the exponent is $\alpha = 1$ a pure capacitor (ideal) behavior

In the limit $d_f \to 3$ the exponent $\alpha \to 0.5$ result of a porous electrode (it is a 3D case)

Observations on Impedance Spectroscopy of super capacitor and di-electric capacitor-states that they are non-ideal and fractional order capacitors

The ideal capacitor is only possible with Mercury Electrode.

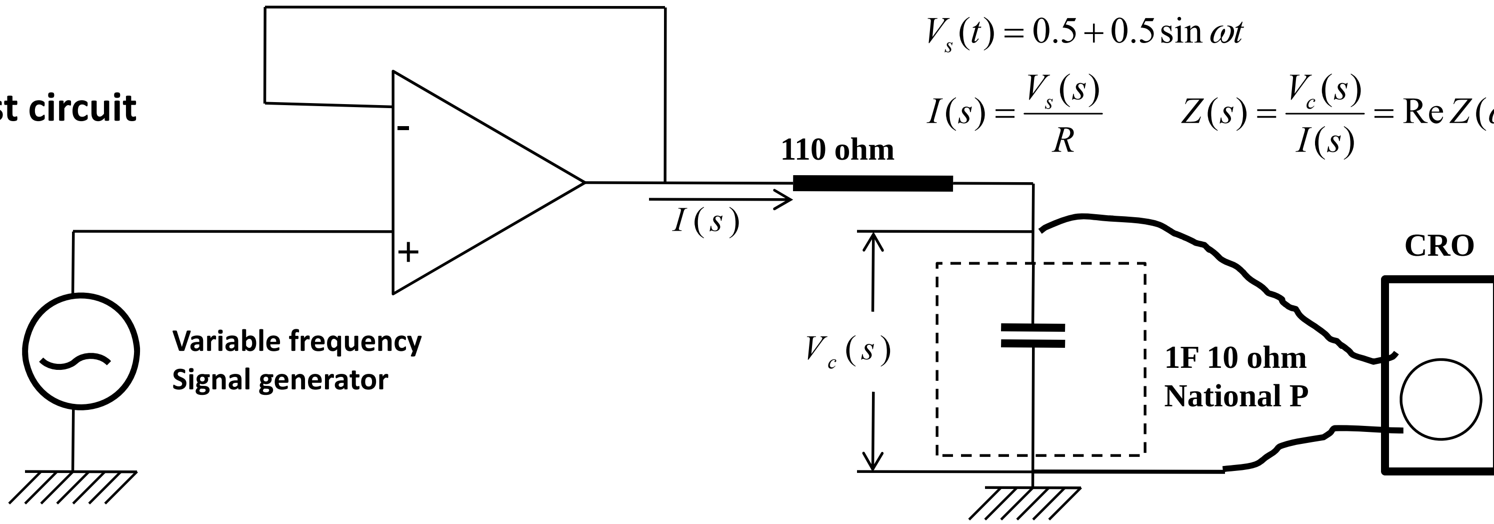
Nyikos and Pajkossy

Impedance plots for market available super-capacitor

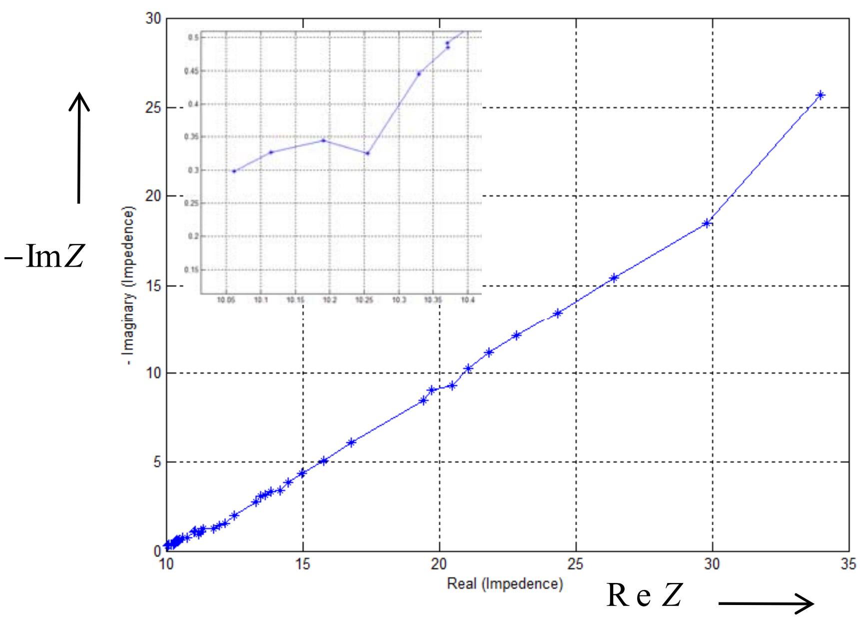
Test circuit

$$V_s(t) = 0.5 + 0.5\sin\omega t$$

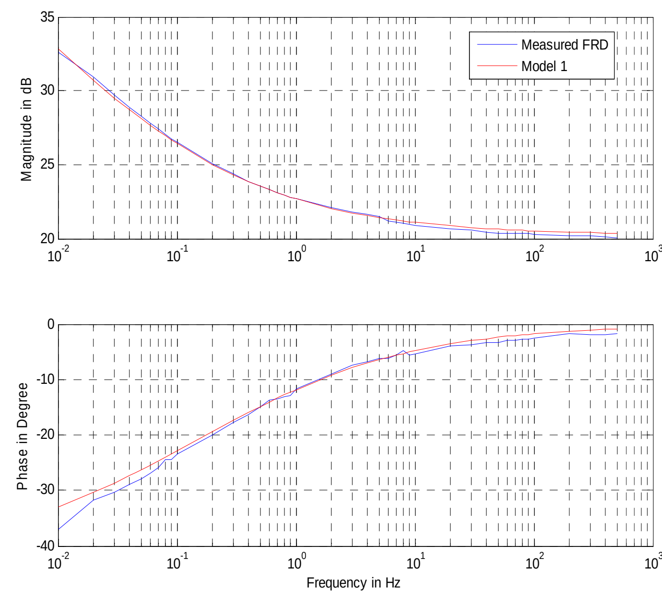
$$I(s) = \frac{V_s(s)}{R} \qquad Z(s) = \frac{V_c(s)}{I(s)} = \mathrm{Re}\,Z(\omega) - \mathrm{j}\,\mathrm{Im}\,Z(\omega)$$



Frequency domain analysis



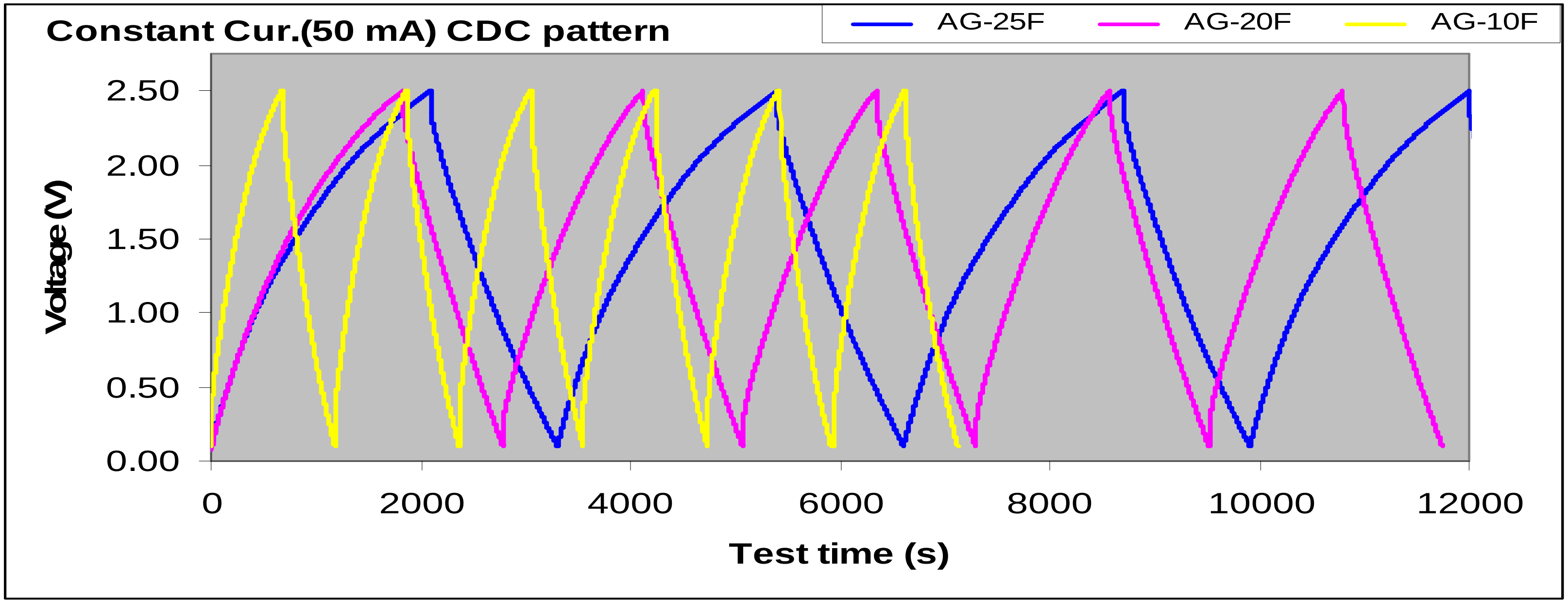
Impedance plot Re (z) and -Im (z) with frequency
Nyquist Plot



Magnitude and Phase Plot of Transfer Function

Under development

Actual observed voltage profile for constant current charge discharge of fractional capacitor-supercapacitor



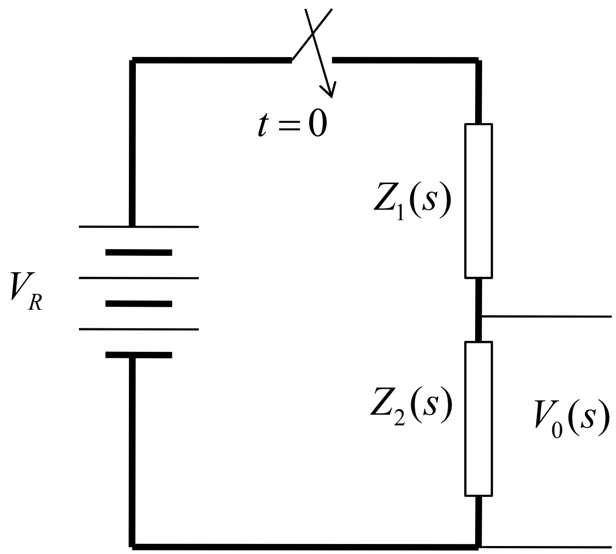
Constant current (50 mA) charge-discharge pattern of 10F, 20 F and 25 F aerogel supercapacitors, studied by using Super Capacitor Test System

Actual observation records non-linear voltage profile of constant current charging and discharging for super-capacitor-gives notion that there is fractional order capacity!!

Courtesy: BRNS Funded joint Project CMET Thrissur-BARC Development of CAG Super-capacitors and application in electronics circuits

Constant voltage charging of fractional capacitor

The figure shows a Low Pass filter where the second impedance is fractional capacitor



$$V_0(s) = \frac{Z_2(s)}{Z_1(s)+Z_2(s)} V_{in}(s)$$

$$Z_2(s) = \frac{1}{s^\alpha C_\alpha}; \quad 0<\alpha<1; \quad Z_1(s) = R$$

$$V_0(s) = \frac{\frac{1}{s^\alpha C_\alpha}}{R+\frac{1}{s^\alpha C_\alpha}} V_{in}(s) = \frac{k}{s^\alpha + k} V_{in}(s), \quad k = \frac{1}{RC_\alpha}$$

$$V_{in}(t) = V_R u(t); \quad u(t) = 1 \quad \text{for} \quad t \ge 0 \quad \text{else} \quad u(t) = 0$$

$$V_{in}(s) = \frac{V_R}{s} \qquad V_0(s) = \frac{V_R k}{s(s^\alpha + k)}$$

$$v_0(t) = \mathcal{L}^{-1}\left\{\frac{V_R k}{s(s^\alpha + k)}\right\}$$

This is voltage charging equation for the fractional capacitor to a constant DC voltage input

Note that we are dealing with fractional differential equation

$$y^{(\alpha)}(t) + k\big(y(t)\big) = k\big(x(t)\big) \qquad 0<\alpha<1$$

$$y(t) = v_0(t) \quad t \ge 0 \qquad y(t) = 0 \quad t<0$$

$$x(t) = v_{in}(t) \qquad v_{in}(t) = V_R \quad t \ge 0 \qquad v_{in}(t) = 0 \quad t<0$$

Solution to obtain Laplace inverse from series

$$V_0(s) = \frac{V_R k}{s(s^\alpha + k)} = \frac{V_R k}{s^{\alpha+1}\left(1 + \frac{k}{s^\alpha}\right)}$$

$$= \frac{V_R k}{s^{\alpha+1}}\left[1 - \frac{k}{s^\alpha} + \frac{k^2}{s^{2\alpha}} - \frac{k^3}{s^{3\alpha}} + \dots\right]$$

$$= V_R\left[\frac{k}{s^{\alpha+1}} - \frac{k^2}{s^{2\alpha+1}} + \frac{k^3}{s^{3\alpha+1}} - \dots\right] \qquad \text{use} \qquad \frac{1}{s^{n+1}} \leftrightarrow \frac{t^n}{\Gamma(n+1)}$$

$$v_0(t) = V_R\left[\frac{kt^\alpha}{\Gamma(\alpha+1)} - \frac{k^2t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k^3t^{3\alpha}}{\Gamma(3\alpha+1)} - \dots\right]$$

$$= V_R\left(1 - \left[1 - \frac{kt^\alpha}{\Gamma(\alpha+1)} + \frac{k^2t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{k^3t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots\right]\right)$$

$$= V_R\left(1 - \sum_{m=0}^{\infty}\frac{(-kt^\alpha)^m}{\Gamma(m\alpha+1)}\right) = V_R\left(1 - E_\alpha(-kt^\alpha)\right)$$

$$= V_R\left[1 - E_\alpha\left(-\frac{t^\alpha}{RC_\alpha}\right)\right]$$

One parameter Mittag-Leffler (1903)function $E_\alpha(x)$

$$E_\alpha(x) \triangleq \sum_{m=0}^{\infty}\frac{(x)^m}{\Gamma(\alpha m+1)}$$

For $\alpha = 1$ we have $E_1(x) = e^x$

Solution using Laplace of Mittag-Leffler function

$$v_0(t) = \mathcal{L}^{-1}\left\{\frac{V_R k}{s(s^\alpha + k)}\right\}$$

We use $\mathcal{L}\left\{t^{\alpha p+\beta-1}E^{(p)}_{\alpha,\beta}(at^\alpha)\right\} = \frac{p!\,s^{\alpha-\beta}}{s^\alpha - a}$ for $p=0$ $\alpha = n$ $\beta = n+1$

to have $\mathcal{L}^{-1}\left\{\frac{s^{-1}}{s^\alpha - a}\right\} = t^\alpha E_{\alpha,\alpha+1}(at^\alpha)$. With this we obtain the following

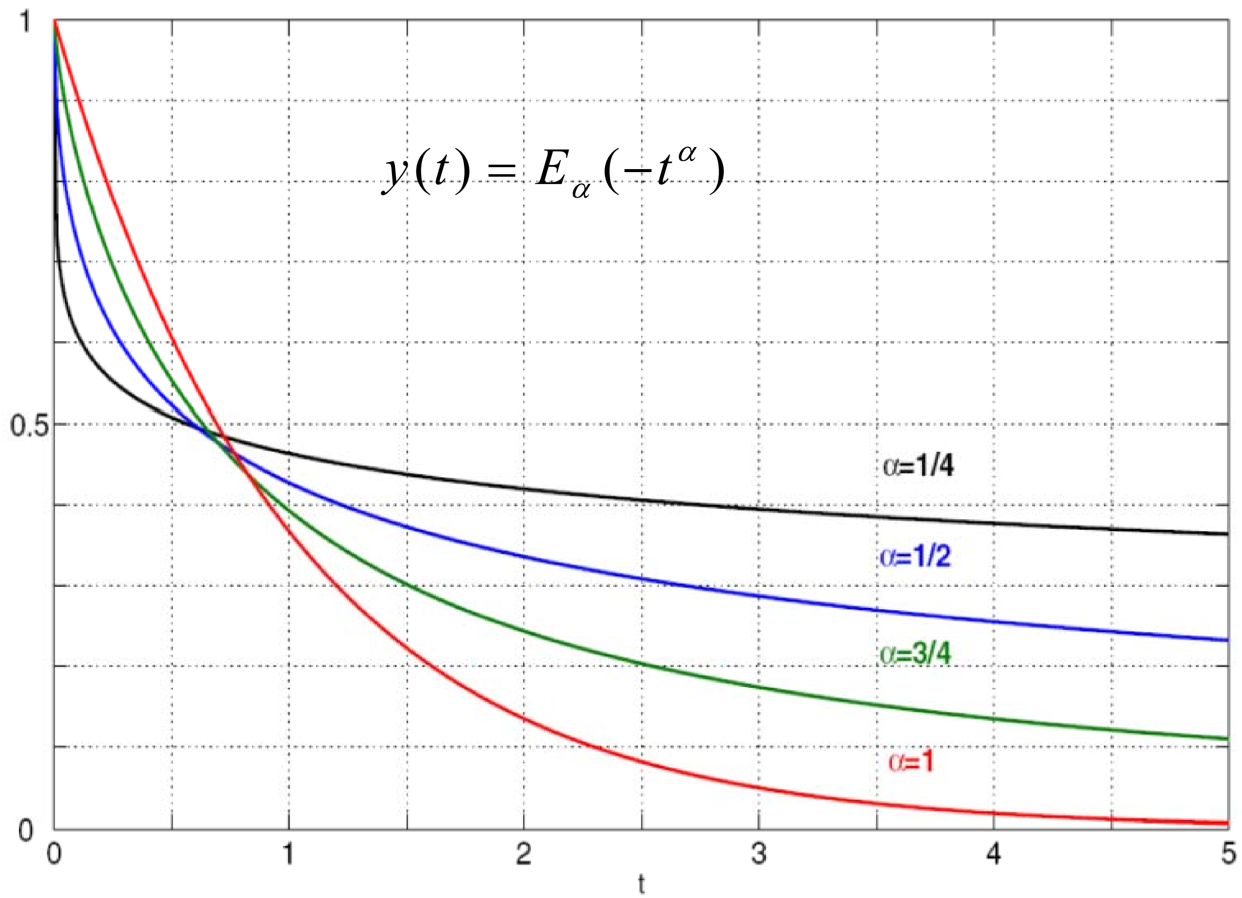
$$v_0(t) = \mathcal{L}^{-1}\left\{\frac{V_R k}{s(s^\alpha + k)}\right\} = V_R k t^\alpha E_{\alpha,\alpha+1}(-kt^\alpha)$$

$$= \frac{V_R}{RC_\alpha} t^\alpha E_{\alpha,\alpha+1}\left(-\tfrac{t^\alpha}{RC_\alpha}\right)$$

Two parameter Mittag-Leffler function (1903) $E_{\alpha,\beta}(x)$ is defined as

$$E_{\alpha,\beta}(x) \triangleq \sum_{m=0}^{\infty} \frac{(x)^m}{\Gamma(\alpha m + \beta)}, \qquad E_{\alpha,(\alpha+1)}(-kt^\alpha) = \sum_{m=0}^{\infty} \frac{(-kt^\alpha)^m}{\Gamma(\alpha m + \alpha + 1)}$$

Graph of decaying Mittag-Leffler function



Mittag-Leffler function plays role in fractional calculus as exponential function is for classical calculus-for solution of fractional differential equation

Charge discharge comparison of classical capacitor and fractional capacitor

For charging voltage of capacitor

$$v_0(t) = V_R\left(1 - E_\alpha\left(-\frac{t^\alpha}{RC_\alpha}\right)\right) = \frac{V_R}{RC_\alpha} t^\alpha E_{\alpha,\alpha+1}\left(-\frac{t^\alpha}{RC_\alpha}\right)$$

For fractional capacitor

For $\alpha = 1$ we get the charging voltage profile of classical ideal capacitor

$$v_0(t) = V_R\left(1 - e^{-\frac{t}{RC}}\right)$$

For charging current of circuit

$$I(s) = \frac{V_R/s}{Z(s)} = \frac{V_R}{s\left(R + \frac{1}{s^\alpha C_\alpha}\right)} = \frac{V_R}{R}\left(\frac{s^{\alpha-1}}{s^\alpha + \frac{1}{RC_\alpha}}\right) \quad \text{using} \quad \mathcal{L}\left\{E_\alpha(at^\alpha)\right\} = \frac{s^{\alpha-1}}{s^\alpha - a}$$

$$i(t) = \frac{V_R}{R} E_\alpha\left(-\frac{t^\alpha}{RC_\alpha}\right)$$

For fractional capacitor

For $\alpha = 1$ we get the charging current profile of classical ideal capacitor

$$i(t) = \frac{V_R}{R} e^{-t/RC}$$

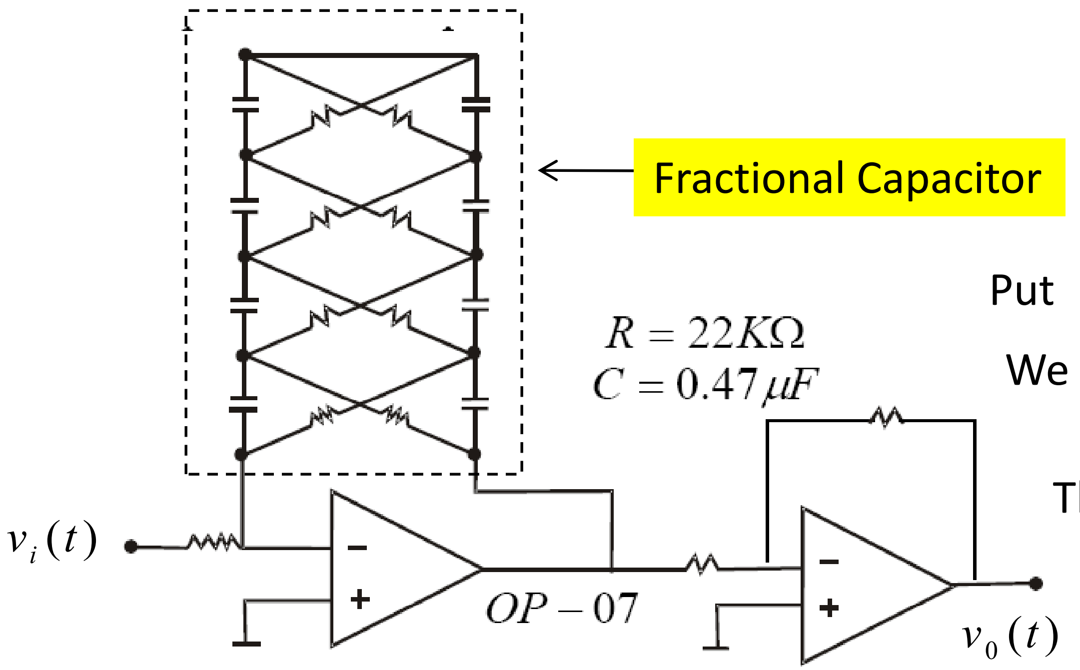
Fractional order capacitance can be realized by $R\ C$ distributed network

The terminal impedance of semi-infinite RC ladder is $Z_f(s) = \sqrt{\frac{R}{C}} \times \frac{1}{s^{1/2}} = \frac{K}{\sqrt{s}}$ $\quad K = \sqrt{\frac{R}{C}}$

The op-amp circuit gives Transfer function as

$$\frac{V_0(s)}{V_i(s)} = \frac{Z_f}{Z_i} = \frac{\sqrt{R\frac{1}{Cs}}}{R} = \frac{K}{\sqrt{s}}$$

$$K = \frac{\sqrt{\frac{22\times10^3}{0.47\times10^{-6}}}}{22\times10^3} = 9.8$$



Put $s = i\omega$ to get $s^{-1/2} = i^{-1/2}\omega^{-1/2}$

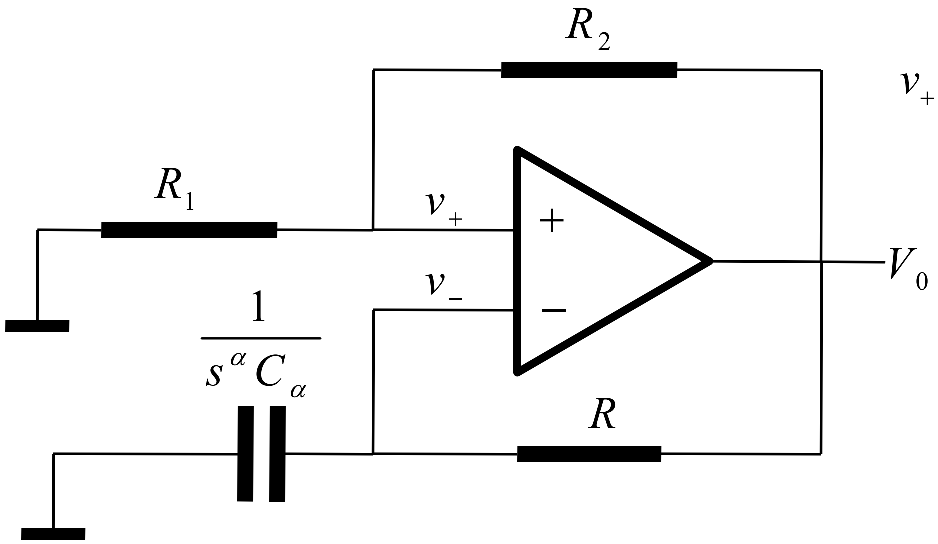
We write $i = e^{i(\pi/2)}$ to get $i^{-1/2} = e^{-i(\pi/4)}$

The transfer function of the circuit is

$$\frac{V_0(\omega)}{V_i(\omega)} = 9.8\omega^{-0.5}e^{-i(\pi/4)}$$

A semi-integrator circuit with $R\ C$ ladder

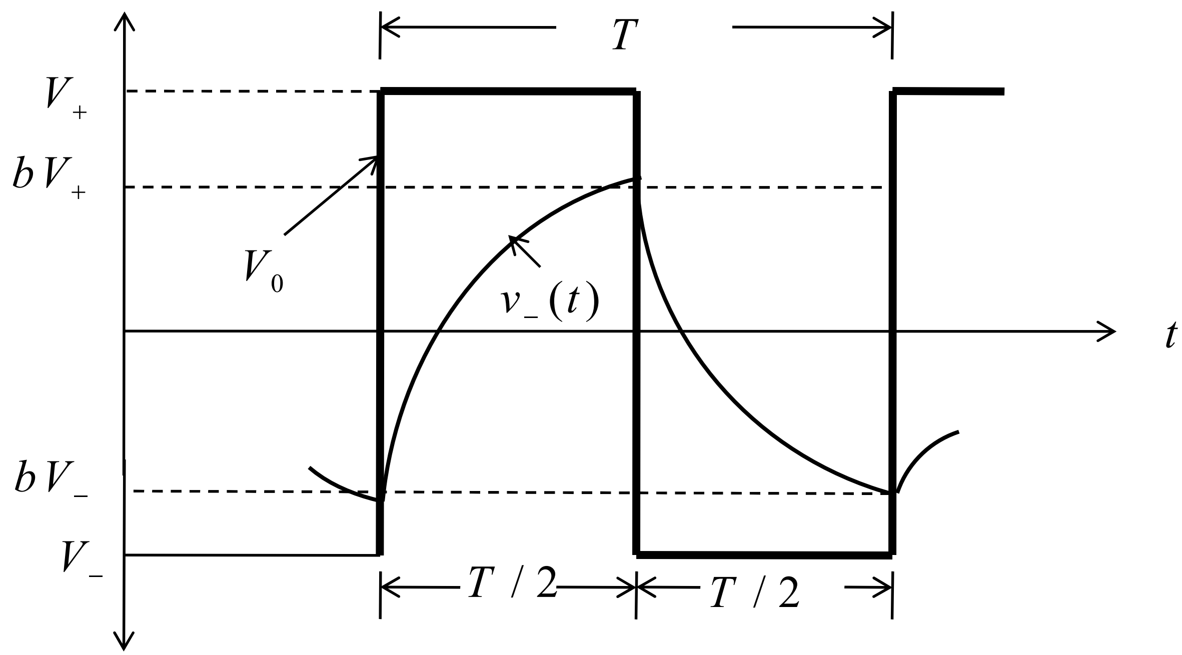
Square Wave Oscillator with Fractional Capacitor-a fractional oscillator



$$v_+ = V_0 \times \frac{R_1}{R_1 + R_2} = bV_0 \qquad b = \frac{R_1}{R_1 + R_2}$$

V_0 output is either V_+ or V_-

the saturation voltage $\pm 15\,\mathrm{V}$



Time period of fractional oscillator

From bV_- to bV_+

$$v_-(t) = V_+\left(1 - E_\alpha\left(-\tfrac{t^\alpha}{RC_\alpha}\right)\right) + (bV_-)E_\alpha\left(-\tfrac{t^\alpha}{RC_\alpha}\right)$$

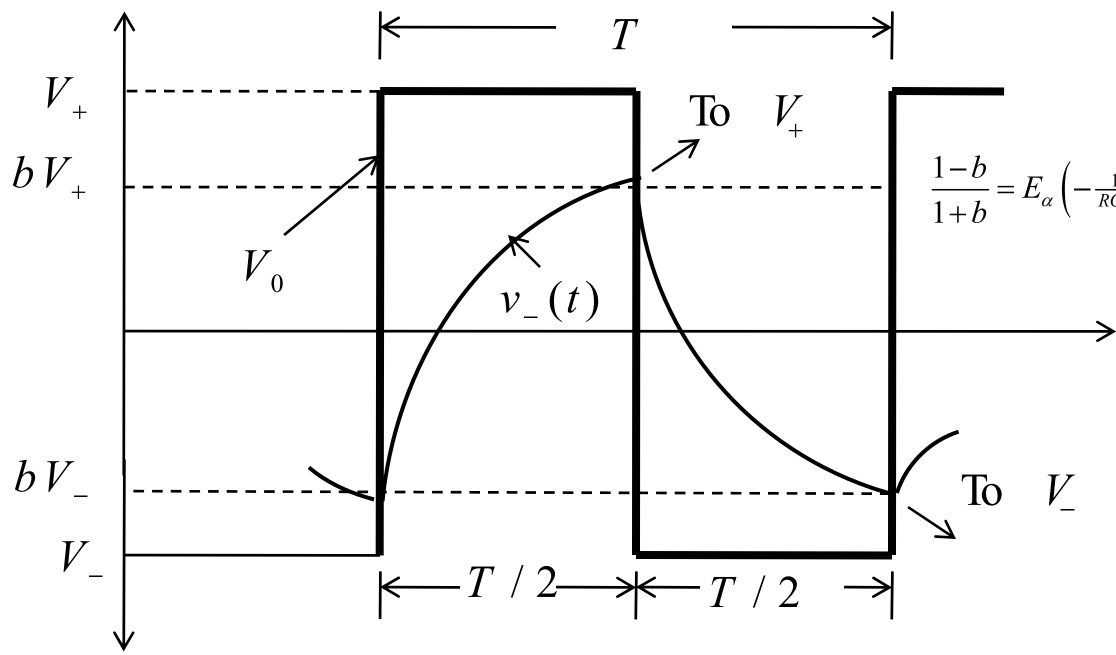
at $\frac{T}{2}$

$$v_-\left(\tfrac{T}{2}\right) = bV_+ = V_+\left(1 - E_\alpha\left(-\tfrac{(T/2)^\alpha}{RC_\alpha}\right)\right) + (bV_-)E_\alpha\left(-\tfrac{(T/2)^\alpha}{RC_\alpha}\right)$$

From bV_+ to bV_-

$$v_-(t) = V_-\left(1 - E_\alpha\left(-\tfrac{t^\alpha}{RC_\alpha}\right)\right) + (bV_+)E_\alpha\left(-\tfrac{t^\alpha}{RC_\alpha}\right)$$

$$v_-\left(\tfrac{T}{2}\right) = bV_- = V_-\left(1 - E_\alpha\left(-\tfrac{(T/2)^\alpha}{RC_\alpha}\right)\right) + (bV_+)E_\alpha\left(-\tfrac{(T/2)^\alpha}{RC_\alpha}\right)$$



$$\frac{1-b}{1+b} = E_\alpha\left(-\tfrac{1}{RC_\alpha}\left(\tfrac{T}{2}\right)^\alpha\right) = \sum_{n=0}^{\infty}\frac{\left(-\tfrac{1}{RC_\alpha}\right)^n\left(\tfrac{T}{2}\right)^{n\alpha}}{\Gamma(n\alpha+1)}$$

$\alpha = 1$

$$\frac{1-b}{1+b} = \exp\left(-\left(\tfrac{T}{2RC}\right)\right)$$

Fractional oscillator oscillates at higher frequency?

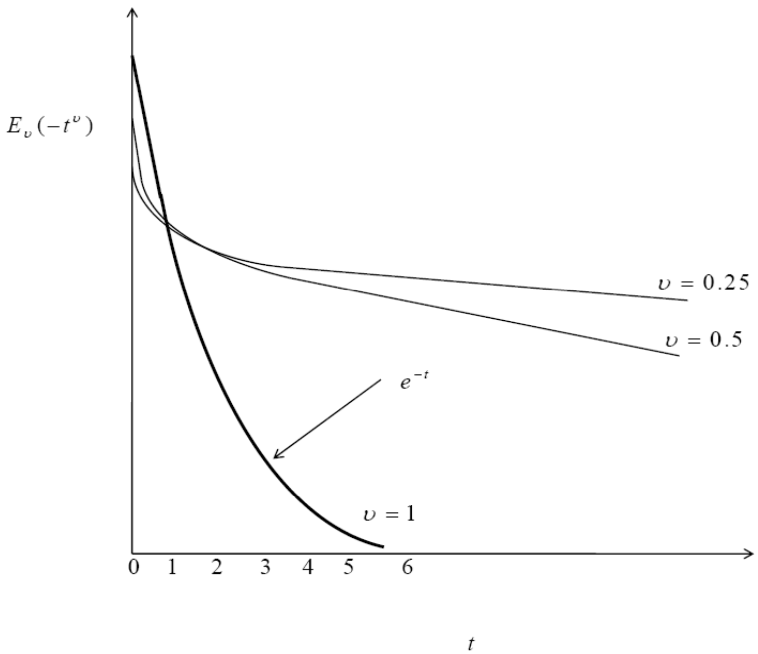
$\alpha = 1$

$C = 1\text{F}, \quad R_s = 7.8\Omega \qquad R = 15\Omega, \quad f = \frac{1}{T} = 0.85\text{Hz}$

$\alpha = 1/2$

$C_\alpha = 0.127\text{F}/\sqrt{\text{s}}, \quad R_s = 7.8\Omega \qquad R = 15\Omega, \quad f = \frac{1}{T} = 11\text{Hz}$

1. Getting about two decades higher frequency? Why?
2. The governing charging discharging function is in fractional order case is Mittag-Leffler function, where for early times, the rise (or fall) is steeper than exponential function.
3. Depends on α and not on C.

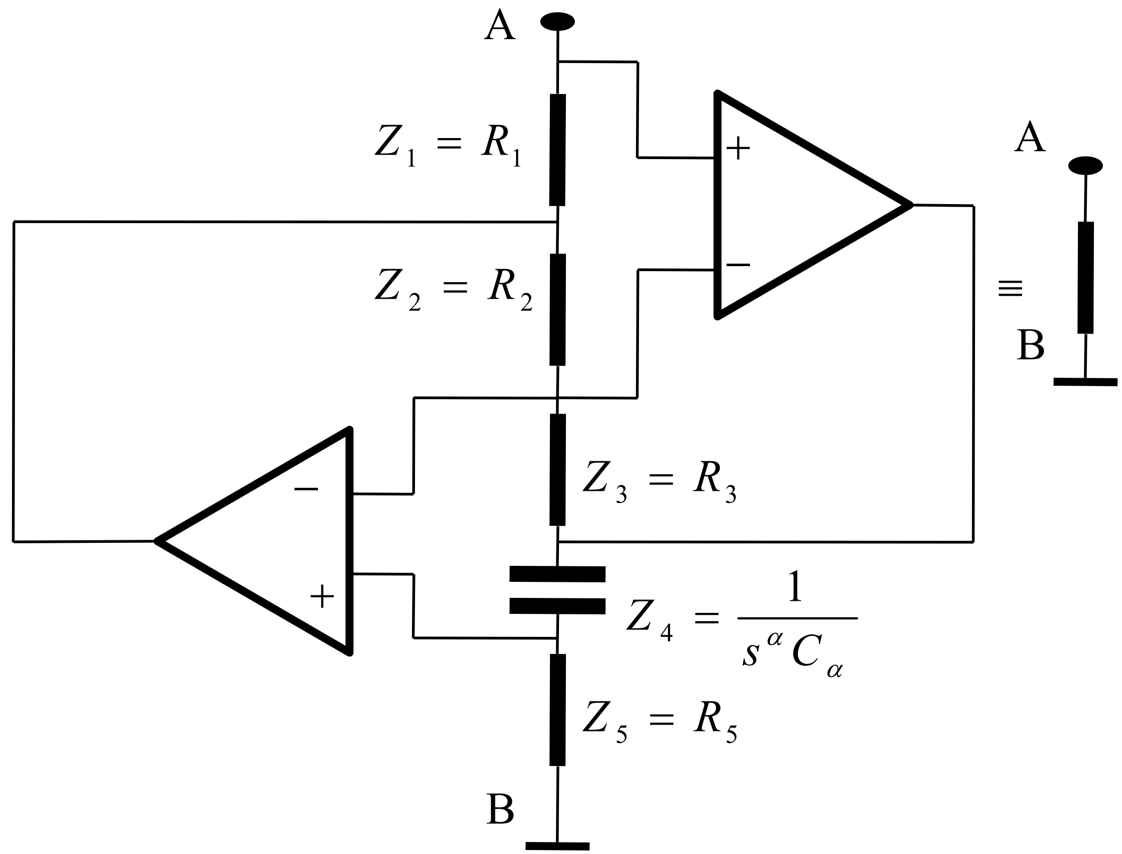


$$\frac{1-b}{1+b} = E_\alpha\left(-\frac{1}{RC_\alpha}\left(\frac{T}{2}\right)^\alpha\right) = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{RC_\alpha}\right)^n \left(\frac{T}{2}\right)^{n\alpha}}{\Gamma(n\alpha+1)}$$

$$\frac{1-b}{1+b} = \exp\left(-\left(\frac{T}{2RC}\right)\right)$$

Fractional Inductor realization

With fractional capacitor implemented in impedance convertor circuit we get fractional inductor



$$Z_{AB} = s^{\alpha} L_{\alpha}$$
$$0 < \alpha < 1$$

$$Z_{AB} = \frac{Z_1 Z_3 Z_5}{Z_2 Z_4} = \frac{s^{\alpha} C_{\alpha} R_1 R_3 R_5}{R_2} = s^{\alpha} L_{\alpha}$$

$$L_{\alpha} = \frac{C_{\alpha} R_1 R_3 R_5}{R_2}$$

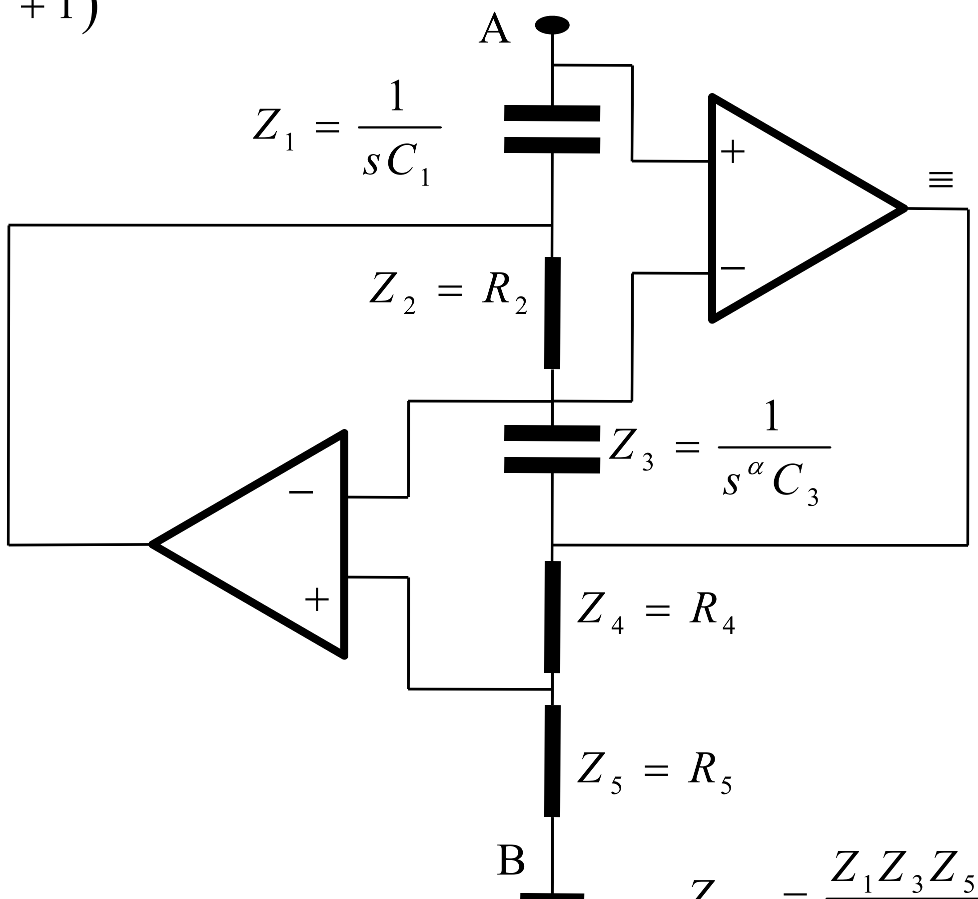
$$V(s) = s^{\alpha} L_{\alpha} I(s)$$

$$v(t) = L_{\alpha} \frac{\mathrm{d}^{\alpha} i(t)}{\mathrm{d} t^{\alpha}}$$

Antoniou Circuit (1969)

Fractional D-element realization

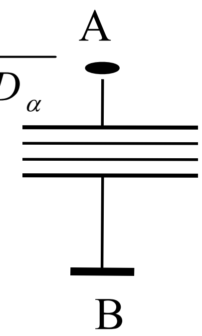
With fractional capacitor implemented in impedance convertor circuit, we get a ‘capacitor’ of order $(\alpha + 1)$



$\equiv$

$$Z_{AB} = \frac{1}{s^{1+\alpha} D_{\alpha}}$$

$$0 < \alpha < 1$$

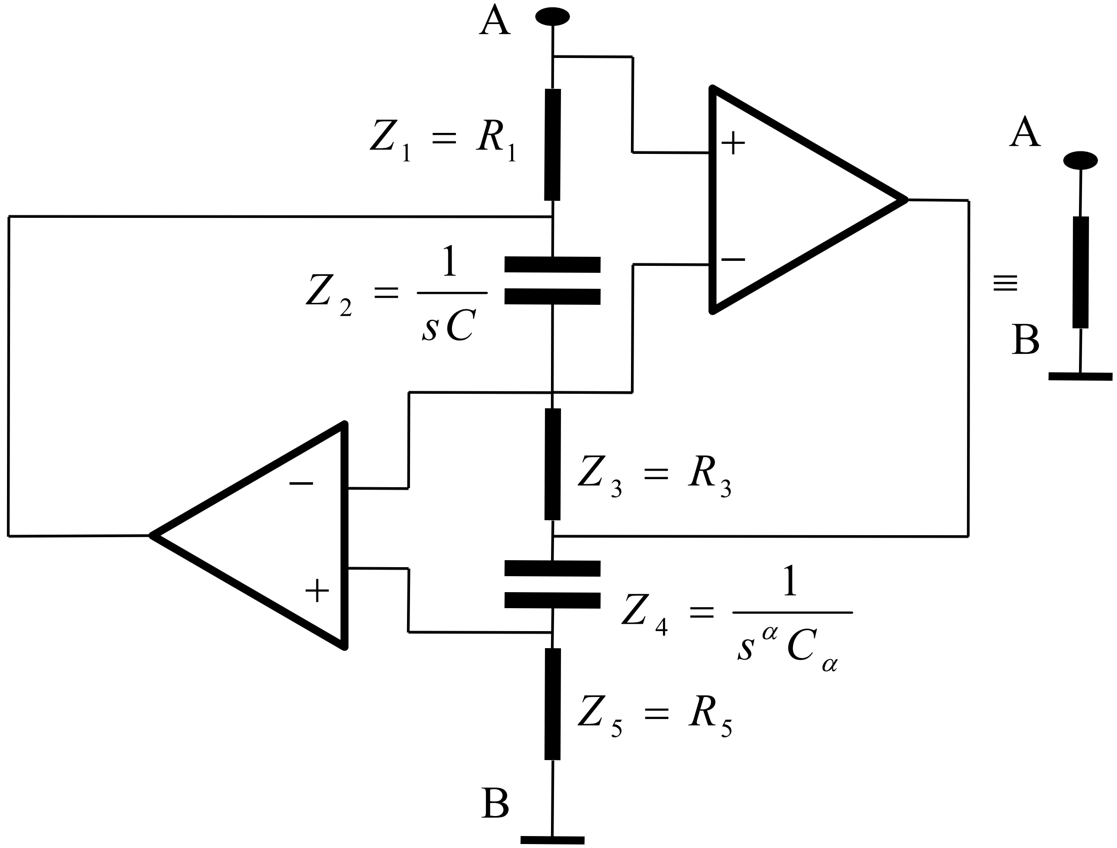


$$Z_{AB} = \frac{Z_1 Z_3 Z_5}{Z_2 Z_4} = \frac{R_5}{s^{\alpha+1} C_1 C_3 R_2 R_4} = \frac{1}{s^{\alpha+1} D_{\alpha+1}}$$

$$D_{\alpha+1} = \frac{C_1 C_3 R_2 R_4}{R_5}$$

Fractional capacitor with order greater than one

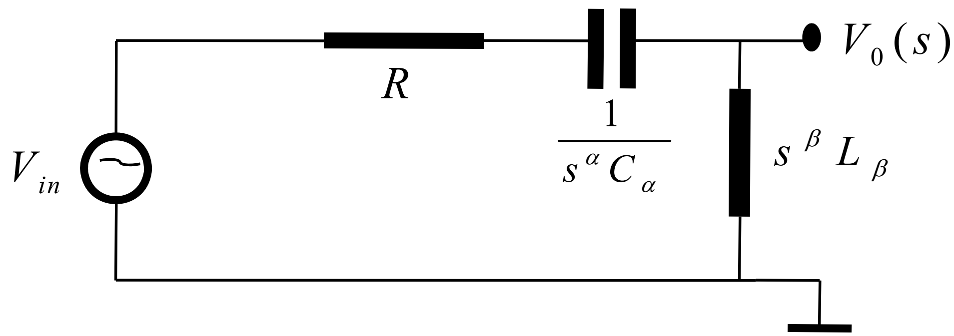
Fractional Inductor with order greater than one realization



$$Z_{AB} = s^{\alpha+1} L_{\alpha+1}$$
$$0 < \alpha < 1$$

$$Z_{AB} = \frac{Z_1 Z_3 Z_5}{Z_2 Z_4} = R_1 (sC) R_3 \left(s^{\alpha} C_{\alpha}\right) R_5 = s^{\alpha+1} L_{\alpha+1}$$
$$L_{\alpha+1} = R_1 R_3 R_5 C C_{\alpha}$$

Fractional Order High Pass Filter



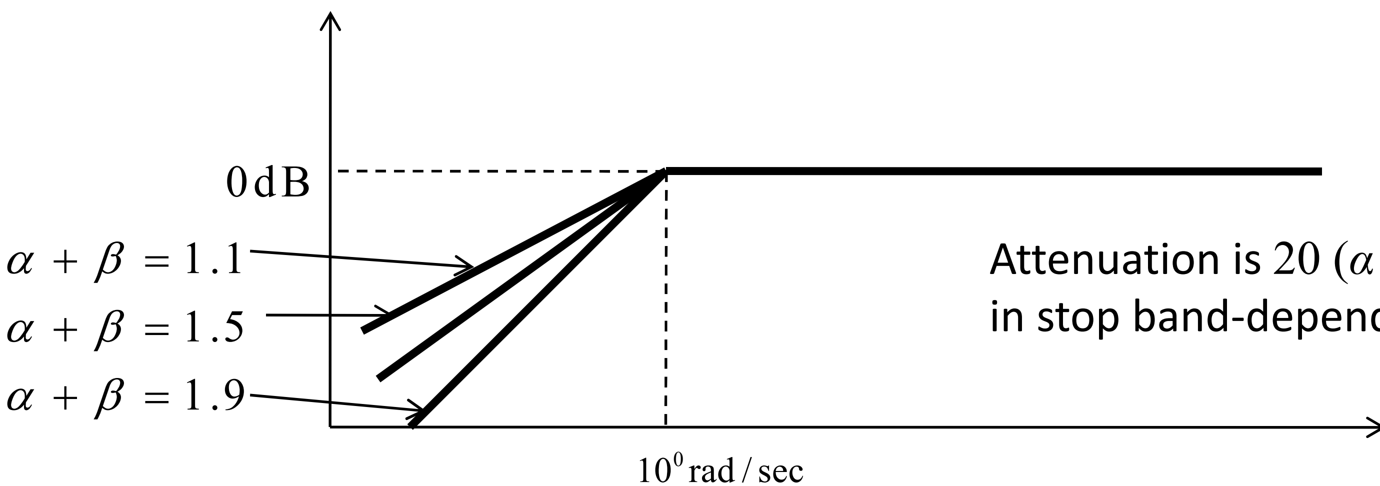
The circuit Transfer Function is

$$G(s) = \frac{V_0(s)}{V_{in}(s)}$$

$$= \frac{s^{\alpha+\beta}}{s^{\alpha+\beta} + s^{\alpha}\left(\frac{R}{L_{\beta}}\right) + \left(\frac{1}{L_{\beta} C_{\alpha}}\right)}$$

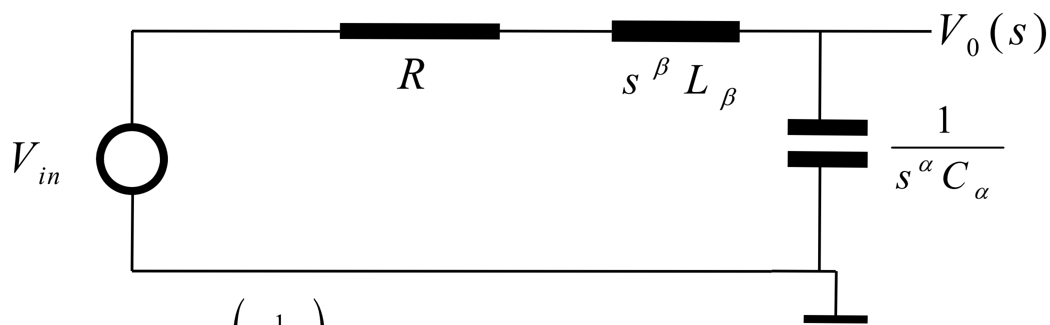
With example values as: $\beta = 1$ $\alpha = 0.1, \quad 0.5, \quad 0.9$ $R = 1, \quad L_{\beta} = 1, \quad C_{\alpha} = 1$

the Bode gain plot is



Attenuation is 20 $(\alpha + \beta)$ dB / decade
in stop band-depends on fractional order

Calculations going into plotting mod of fractional order transfer function for a fractional order low pass filter (FOLPF) -for Bode Gain Magnitude Plot



$$G_{\text{FOLPF}}(s) = \frac{\left(\frac{1}{L_\beta C_\alpha}\right)}{s^{\alpha+\beta} + s^{\beta}\left(\frac{R}{L_\beta}\right) + \left(\frac{1}{L_\beta C_\alpha}\right)} \qquad \text{put} \qquad s = i\omega \qquad i = \cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2}$$

Denominator $\quad s^{\alpha+\beta} + s^{\alpha}\left(\frac{R}{L_\beta}\right) + \left(\frac{1}{L_\beta C_\alpha}\right) = (i\omega)^{\alpha+\beta} + (i\omega)^{\alpha}\left(\frac{R}{L_\beta}\right) + \left(\frac{1}{L_\beta C_\alpha}\right)$

$$= \omega^{\alpha+\beta}\left(\cos\tfrac{(\alpha+\beta)\pi}{2} + i\sin\tfrac{(\alpha+\beta)\pi}{2}\right) + \omega^{\alpha}\left(\tfrac{R}{L_\beta}\right)\left(\cos\tfrac{\alpha\pi}{2} + i\sin\tfrac{\alpha\pi}{2}\right) + \tfrac{1}{L_\beta C_\alpha}$$

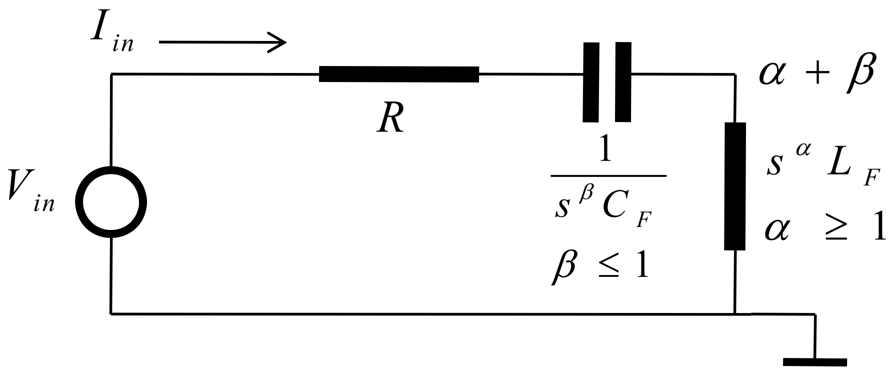
$$= \left(\tfrac{1}{L_\beta C_\alpha} + \tfrac{R}{L_\beta}\omega^{\alpha}\cos\tfrac{\alpha\pi}{2} + \omega^{\alpha+\beta}\cos\tfrac{(\alpha+\beta)\pi}{2}\right) + i\left(\sin\tfrac{(\alpha+\beta)\pi}{2} + \sin\tfrac{\alpha\pi}{2}\right) = u + iv$$

use $\quad |u+iv| = \sqrt{u^2+v^2} \quad$ and $\quad \angle u+iv = \tan^{-1}\left(\frac{v}{u}\right) \quad$ simplify to get

$$\left|G_{\text{FOLPF}}(\omega)\right| = \left(2RLC_\alpha^{\;2}\omega^{2\alpha+\beta}\cos\tfrac{\beta\pi}{2} + 2RC_\alpha\omega^{\alpha}\cos\tfrac{\alpha\pi}{2} + 2L_\beta C_\alpha\omega^{\alpha+\beta}\cos\tfrac{(\alpha+\beta)\pi}{2} + R^2C_\alpha^{\;2}\omega^{2\alpha} + L_\beta^{\;2}C_\alpha^{\;2}\omega^{2(\alpha+\beta)} + 1\right)^{-1/2}$$

$$\angle G_{\text{FOLPF}}(\omega) = \tan^{-1}\left(\frac{\sin\frac{(\alpha+\beta)\pi}{2} + \sin\frac{\alpha\pi}{2}}{\frac{1}{L_\beta C_\alpha} + \frac{R}{L_\beta}\omega^{\alpha}\cos\frac{\alpha\pi}{2} + \omega^{\alpha+\beta}\cos\frac{(\alpha+\beta)\pi}{2}}\right)$$

Fractional Order Band Pass Filter gives



$\alpha + \beta \le 2$

Admittance

$$Y(s) = \frac{I_{in}(s)}{V_{in}(s)}$$

$$= \frac{\frac{s^{\beta}}{L_F}}{s^{\alpha+\beta} + \left(\frac{R}{L_F}\right)s^{\beta} + \left(\frac{1}{C_F L_F}\right)}$$

$$Y_{\mathrm{Re}} = \frac{1}{R + L_F \omega^{\alpha} \cos\left(\frac{\pi\alpha}{2}\right) + \left(\frac{1}{C_F \omega^{\beta}}\right)\cos\left(\frac{\pi\beta}{2}\right)} \quad \text{at} \quad \omega_{\mathrm{Re}} = \left(\frac{\sin\left(\frac{\pi\beta}{2}\right)}{L_F C_F \sin\left(\frac{\pi\alpha}{2}\right)}\right)^{\frac{1}{(\alpha+\beta)}}$$

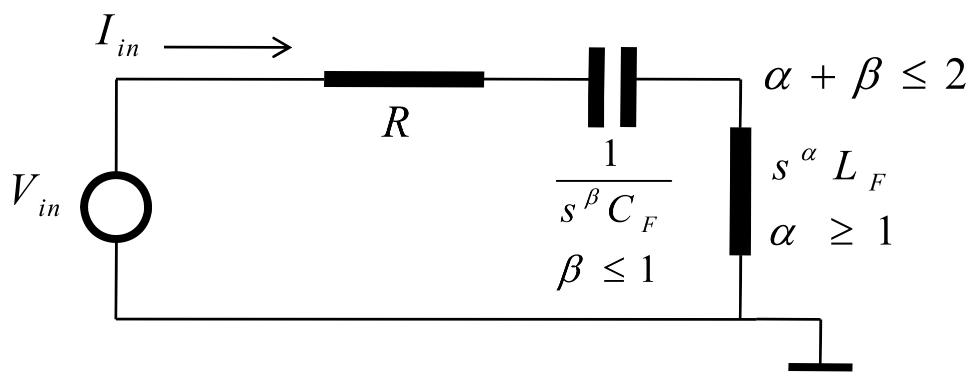
We observe that the real admittance is a function of frequency !!

Whereas in classical case the real admittance is $Y_{\mathrm{Re}} = \frac{1}{R}$ at $\omega_{\mathrm{Re}} = \sqrt{\frac{1}{L_F C_F}}$

Int. J. Circ. Theory Appl. (2014): Experimental studies on realization of Fractional Inductors & Fractional Order Band Pass Filters

Courtesy :Dept. of EE IIT Kgp

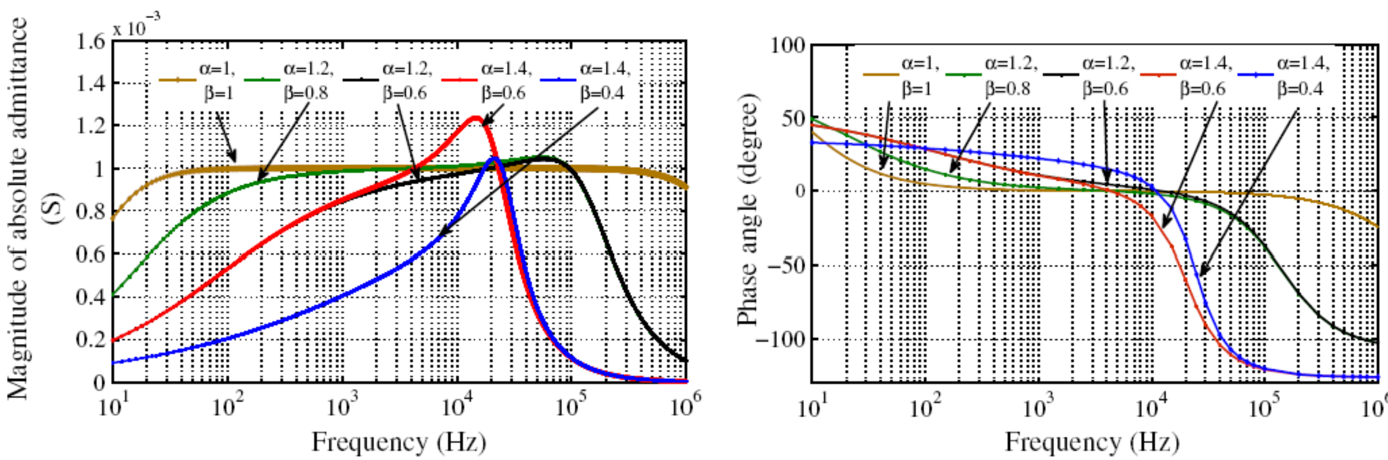
Fractional Order Band Pass Filter gives better tuning !!



$$R = 1\text{K}$$

$$C_F = 1.89 \times 10^{-5}$$

$$L_F = 7.2 \times 10^{-5}$$



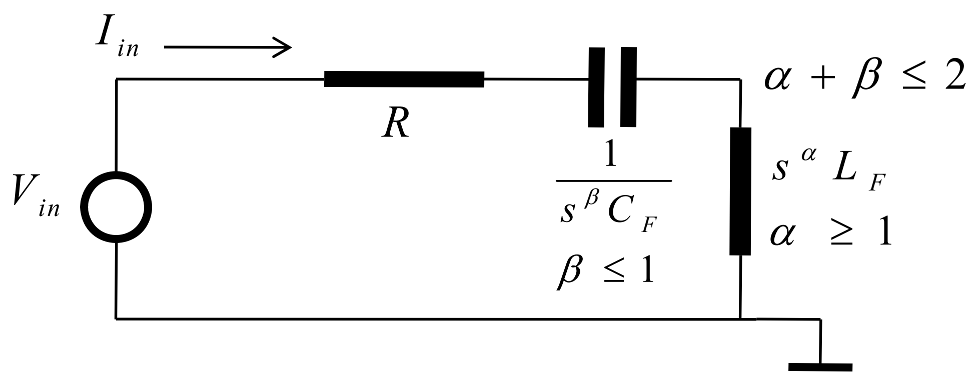
(a) Magnitude of absolute admittance (b) Phase of absolute admittance

Get sharper tuning characteristics and narrower Band-width by choice of fractional orders
Frequency at which the Admittance magnitude is maximum do not coincide with the frequency where phase Angle is zero.

Int. J. Circ. Theory Appl. (2014): Experimental studies on realization of Fractional Inductors & Fractional Order Band Pass Filters

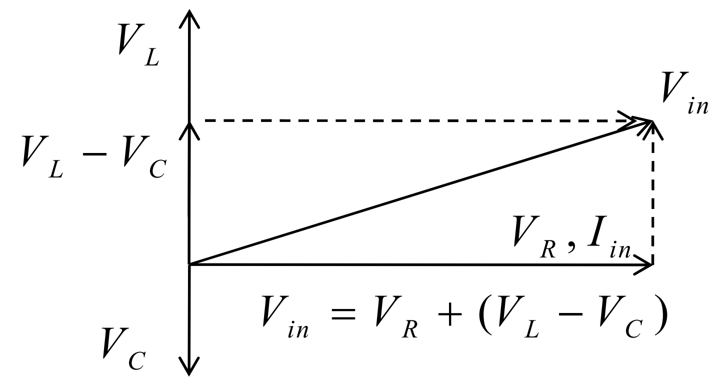
Courtesy :Dept. of EE IIT Kgp.

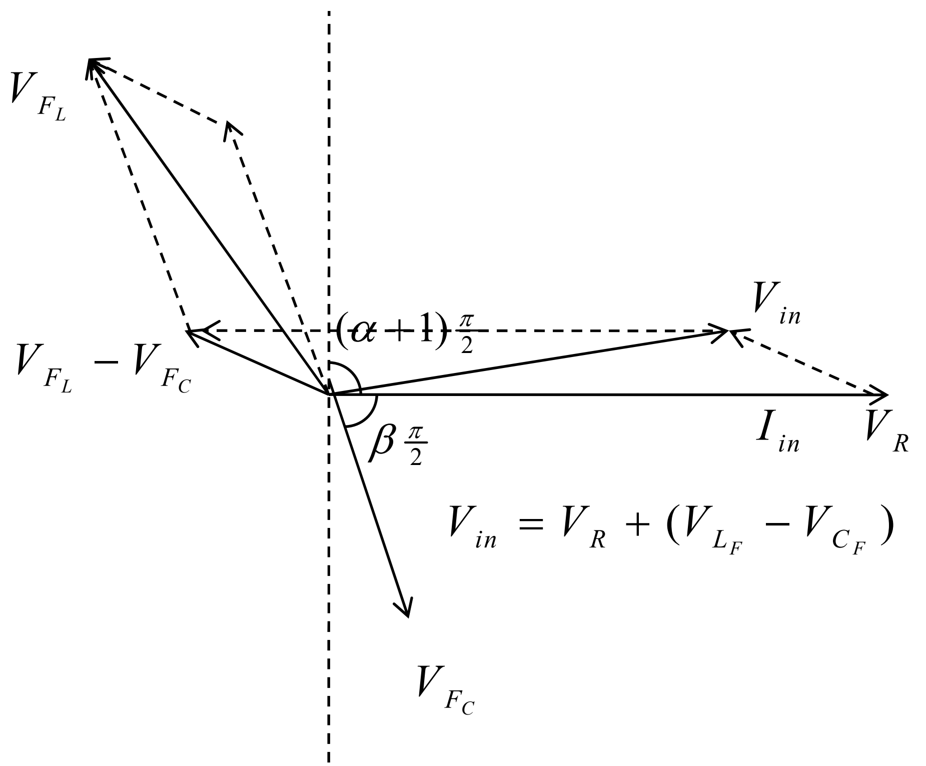
Fractional Order Resonance



For $\alpha = 1; \quad \beta = 1$

The phasor diagram is for R-L-C circuit





For $\alpha > 1; \quad \beta < 1$

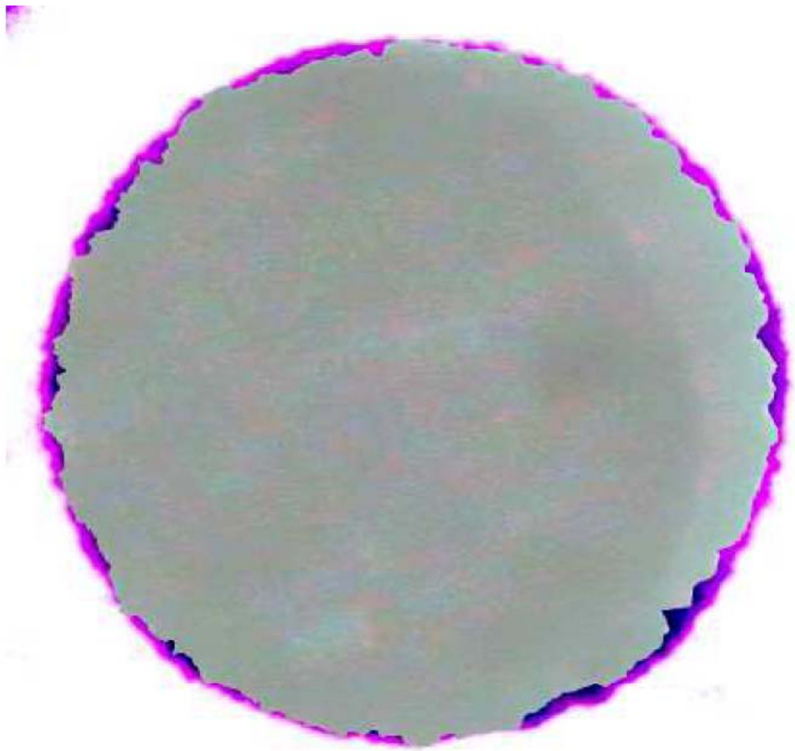
L_F, and C_F together contribute to provide a negative resistance

Effective resistance decreases

Admittance increases

Q-factor of fractional resonance circuit with inductor order greater than one is better Than normal R-L-C circuit.

Visco-elastic experiments on several starch samples



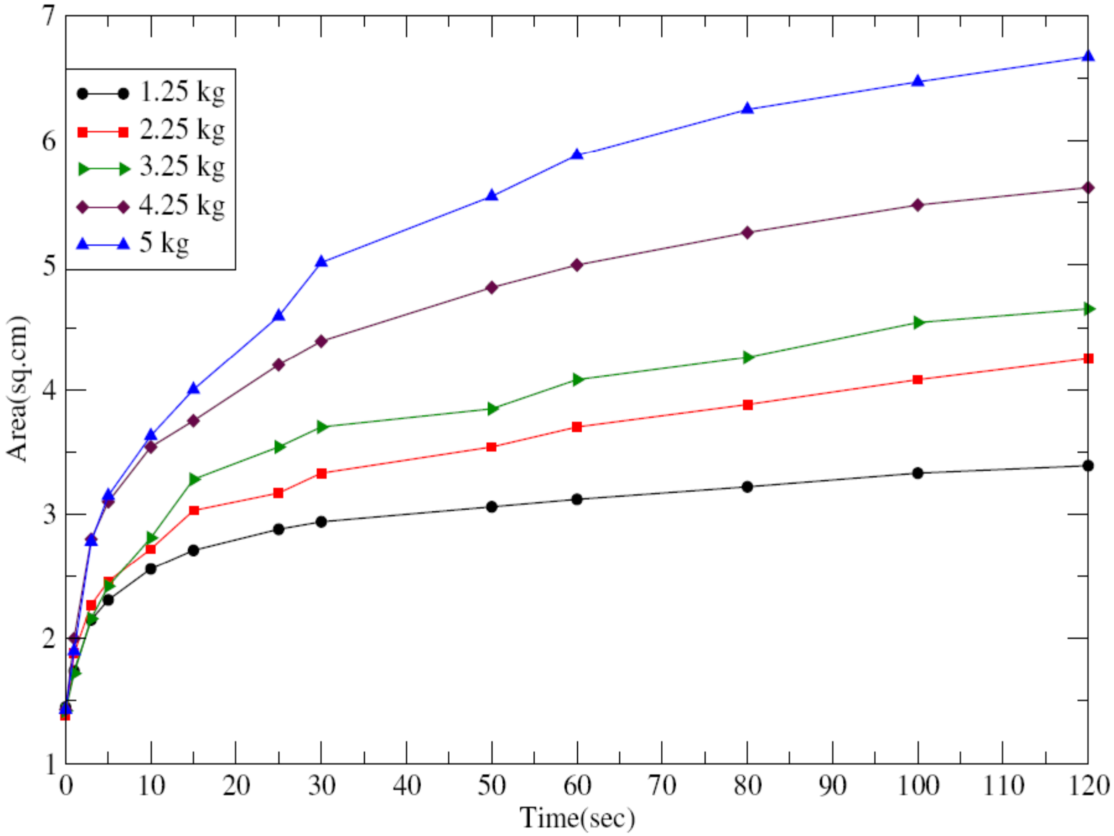
A snapshot of the film (inner blob) superposed on the photograph of the film photograph of the film about 2 sec. earlier (outline visible along the periphery) shows the shrinking of the film-'oscillatory spreading of area (strain) of the starch under Load (stress)'

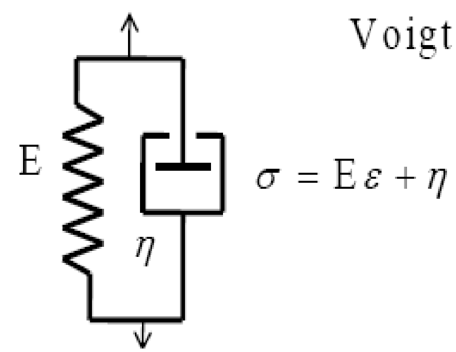
Spreading of Non-Newtonian & Newtonian Fluids on a Solid Surface under Pressure, J of Phys: Conf Series 319(2011)

Forced Spreading & Rheology of Starch Gel: Viscoelastic modeling with Fractional Calculus : Colloids & Surfaces A : Physicochemical & Engineering Aspects 407 (2012) 64-70

Courtesy : CMRC Dept. of Phys JU

Spreading of Newtonian Fluid





$$\sigma = E\varepsilon + \eta \frac{d\varepsilon}{dt}$$

$$\beta \frac{d}{dt}\varepsilon(t) + E\varepsilon(t) = \sigma(t)$$

An area-time plot (castor oil on perspex)

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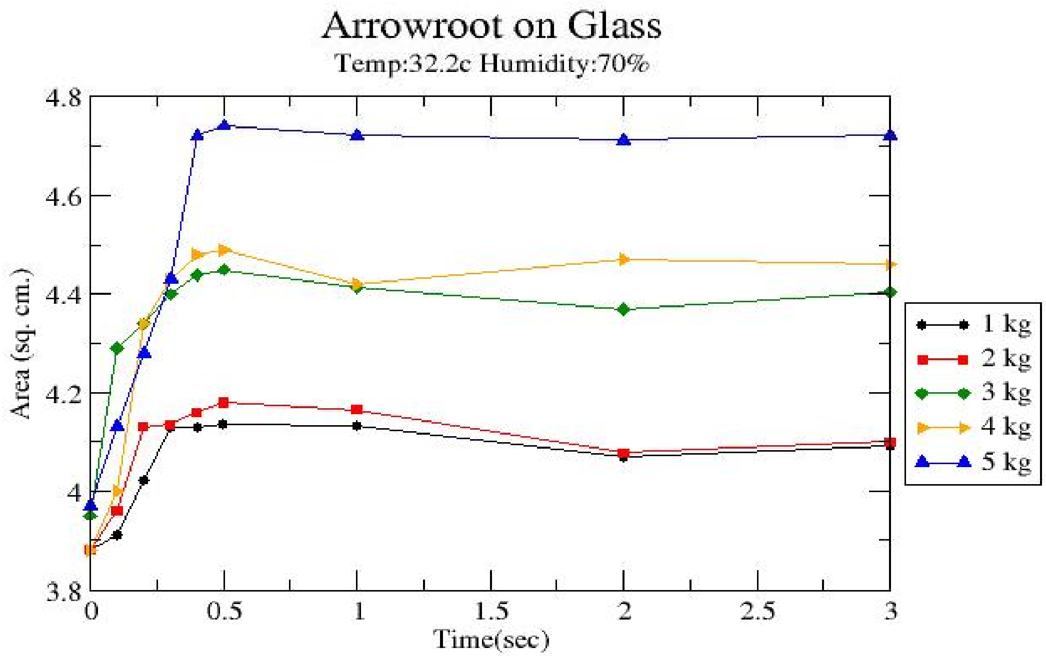
Courtesy : CMRC Dept. of Phys JU

Spreading of Non-Newtonian Fluid-gives a oscillatory spreading



An area-time plot (Arrowroot on Glass))

The non-Newtonian area-time plot $\frac{d^{\alpha}}{dt^{\alpha}}\varepsilon(t) + B\varepsilon(t) = \frac{1}{\beta}\sigma(t)$ $\quad B = E/\beta$

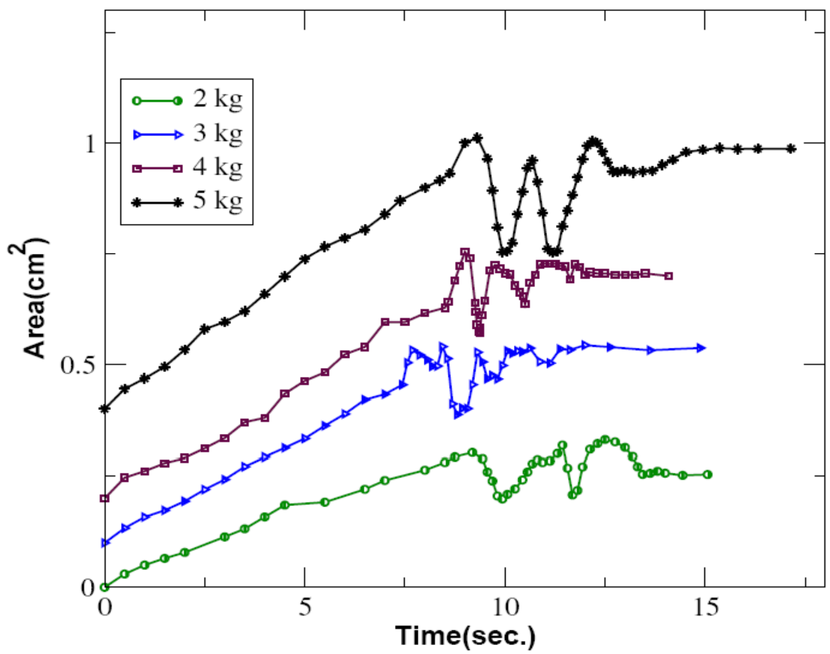
With $\alpha = 1.1$ the integer model gets generalized to fractional differential Equation Fractional Voigt Model!

Spreading of Non-Newtonian & Newtonian Fluids on a Solid Surface under Pressure, J of Phys: Conf Series 319(2011)

Forced Spreading & Rheology of Starch Gel: Viscoelastic modeling with Fractional Calculus : Colloids & Surfaces A : Physicochemical & Engineering Aspects 407 (2012) 64-70

Courtesy : CMRC Dept. of Phys JU

Oscillatory Spreading of Non-Newtonian Fluid



An area-time plot (Potato starch gel on glass)

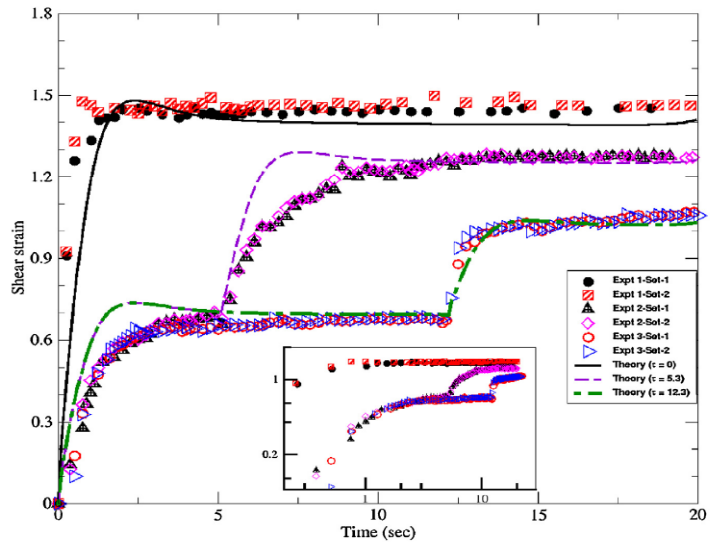
Experimental variation in area with time for different values of load for 2.5 concentration potato starch gel on glass

Spreading of Non-Newtonian & Newtonian Fluids on a Solid Surface under Pressure, J of Phys: Conf Series 319(2011)

Forced Spreading & Rheology of Starch Gel: Viscoelastic modeling with Fractional Calculus : Colloids & Surfaces A : Physicochemical & Engineering Aspects 407 (2012) 64-70

Courtesy : CMRC Dept. of Phys JU

Effect of loading history on viscoelastic property of a non-Newtonian fluid: analysis using fractional calculus



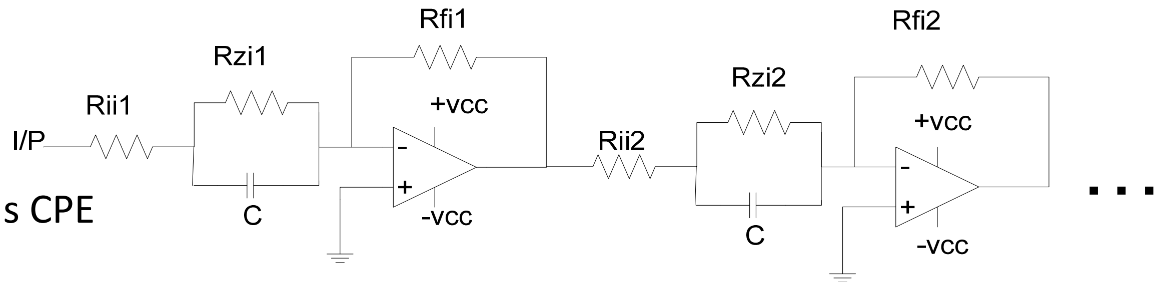
An area-time plot (Potato starch gel spreading)

In this work gelatinized starch is shown to retain the memory of past loading history. It exhibits a visco-elastic response which does not depend solely on instantaneous conditions. A simple squeeze flow experiment is performed, where loading is done in two steps with a time lag τ seconds between the steps. The effect on the strain, of varying τ is reproduced by a visco-elastic model. Complexity is introduced through Fractional Calculus by incorporating non-integer derivatives in viscosity equations. A strain-hardening proportional to time lag between two steps is also incorporated. This model reproduces salient features observed in experiment namely-memory effect, slight initial oscillations in strain as well as long time solid like response.

Effect of loading history on visco-elastic potato starch gel: Colloids & Surfaces A: Physicochemical & Engineering Aspect 492 (2016) 47-53

Courtesy : CMRC Dept. of Phys JU

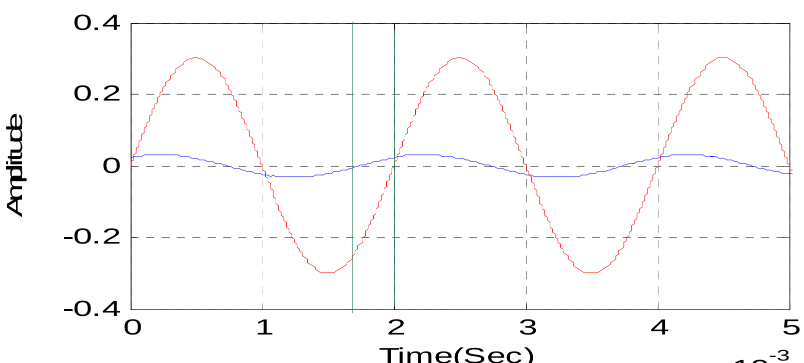
Realized active circuit for fractional differ-integration

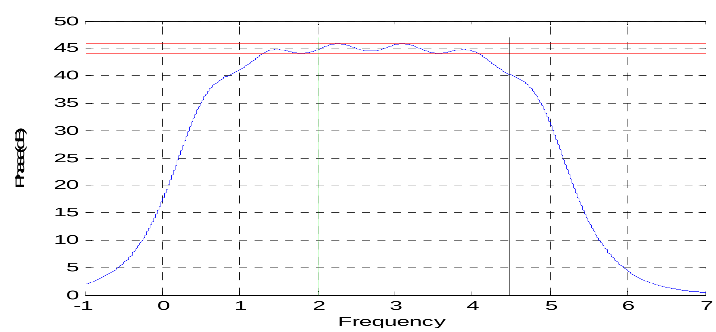


Making the Bode phase plot flat as CPE

$$s^{1/2} \cong \frac{(s - z_1)(s - z_2)..}{(s - p_1)(s - p_2)..}$$

i	Zi	Pi	Ci	Rfi= Rii		Rzi	
				Ω	TP	Ω	TP
1	2.2537	6.0406	1μ	264.07k	500k	443.71k	500k
2	15.955	42.764	1μ	37.30k	50k	62.67k	100k
3	112.95	302.75	680nf	11.21k	20k	18.83k	20k
4	799.65	2143.3	68nF	10.94k	20k	18.39k	20k
5	5661.1	15173	10nF	10.51k	20k	17.64	20k
6	40078	107420	1nF	14.85k	20k	24.95k	50k

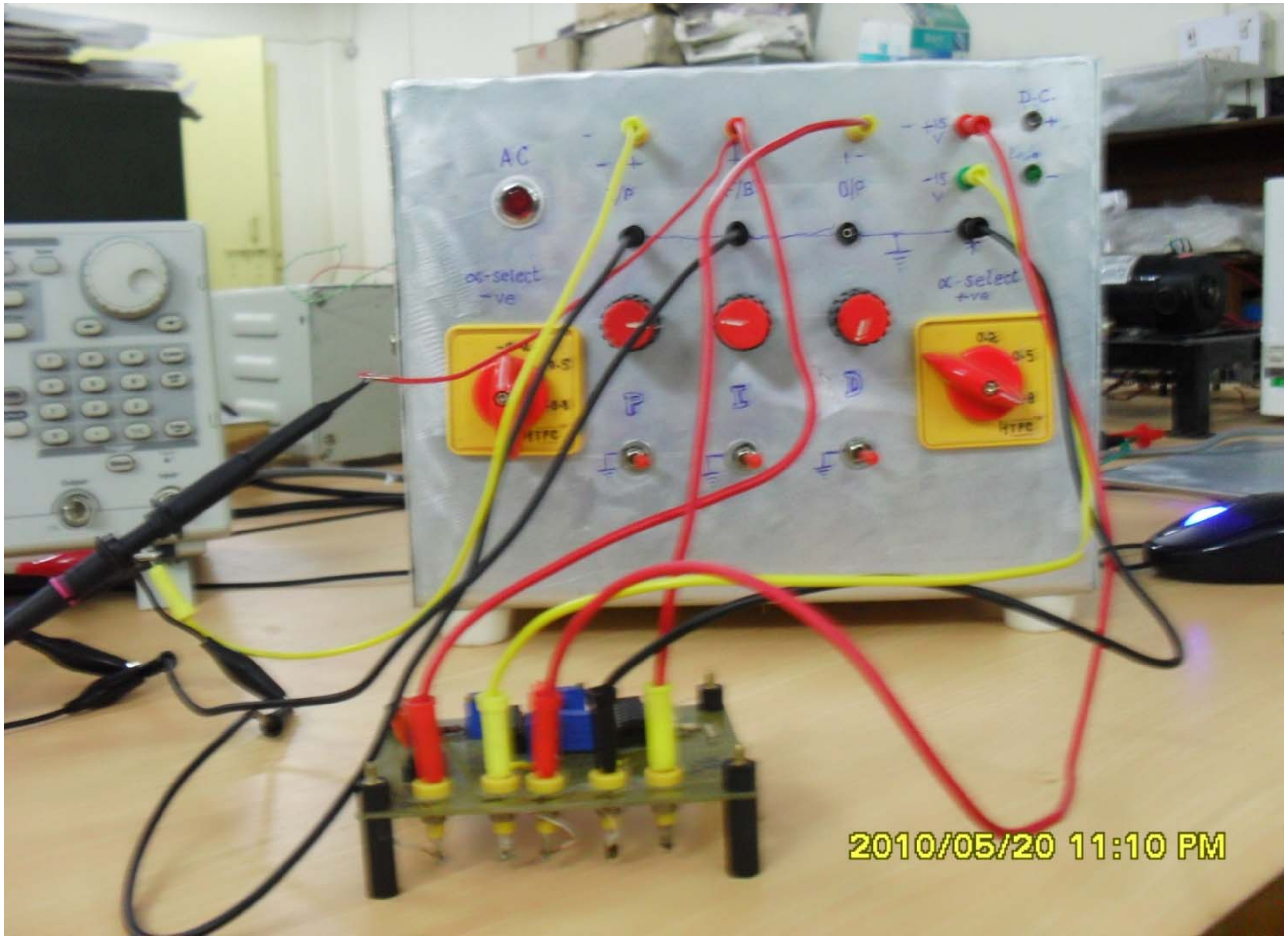




A new method for getting rational approximation for Fractional Order Differintegrals :Asian j of Controls Vol 18, No 4 Pp 1-18.

Courtesy VNIT-Nagpur Dept of EE

Fractional Order Circuits and Systems Hardwired FO-PID



Hardwired Fractional Order PID connected to DC Motor Emulator Circuit

Courtesy VNIT-Nagpur Dept of EE

Coarse graining phenomenon and fractional calculus

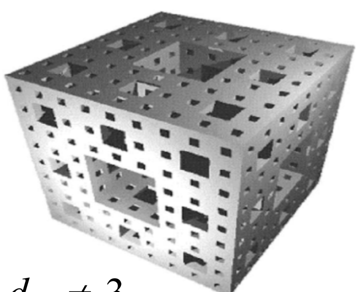
In systems involving course grained phenomenon, every thing happens as if the elemental point is not infinitesimally small (zero) rather it exhibits some thickness, what could be pictured by using $(\mathrm{d}x)^\alpha$ $0 < \alpha < 1$, instead of $\mathrm{d}x$

As $\mathrm{d}x \downarrow 0$ we will have always $(\mathrm{d}x)^\alpha > (\mathrm{d}x)$

In other words we are considering rate of variation as $\dfrac{\mathrm{d}(f(x))}{(\mathrm{d}x)^\alpha}$ or $\dfrac{\mathrm{d}^\alpha(f(x))}{(\mathrm{d}x)^\alpha}$

Here we come across fractional derivative and fractional calculus

When media is non-uniform if we shrink the element to zero, then we loose its actual picture



$d_f \neq 3$

$$\mathrm{div}\,\mathbf{J} \triangleq \lim_{V \to 0} \frac{1}{V} \oint_S \mathbf{J} \bullet \mathbf{n}\, \mathrm{d}S \equiv \nabla \bullet \mathbf{J}$$

$$\mathrm{div}^\beta\,\mathbf{J} \triangleq \lim_{V \to \mathrm{REV}} \frac{1}{V} \oint_S \mathbf{J} \bullet \mathbf{n}\, \mathrm{d}S \equiv \nabla^\beta \bullet \mathbf{J}$$

REV is Representative Elementary Volume and a non-zero

Concept of fractional divergence

What we should ask ourselves

Calculus is as old as three hundred plus years so is fractional calculus

Why we need to have fractional calculus ?

The differentiation $\frac{\mathrm{d}}{\mathrm{d}x}$ operation is taking rate per unit (infinitesimal) $\mathrm{d}x$ as $\mathrm{d}x \downarrow 0$

For an element $(\mathrm{d}x)^{\alpha}$ defining rate as $\mathrm{d}x \downarrow 0$ for say $0 < \alpha < 1$ gives notion of fractional differentiation

Similarly integration with respect to element $(\mathrm{d}x)^{\alpha}$ gives notion of fractional integration

Obviously $(\mathrm{d}x)^{\alpha} > \mathrm{d}x$ what does it say then ? That is coarse graining-views roughness/non-uniformity

Fractional calculus is extension of the classical calculus

Is there any conjugation or parallel ?

Can we apply to non-differentiable functions?

System dynamics having memory has direct link with fractional calculus

We should revisit the physical units in case we are considering fractional rate while working with fractional differentials; like $\mathrm{F}/\mathrm{s}^{1-\alpha}$ instead conventional F-and their physical interpretation.

A paradox of mathematicians now a reality in science & engineering

Still several miles to go to understand

"Which Calculus the Nature obeys ?"

……….yet a developing subject………

THANKYOU ALL