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INFN - Roma Tre

IX International Workshop on
CKM Unitarity Triangle
TIFR, Mumbai, India,
November 28 - December 2, 2016

V_{us} from kaon decays in theory

outline of the talk

* ***leptonic $K_{\ell 2}$ and $\pi_{\ell 2}$ decays***: V_{us} / V_{ud} from lattice determinations of f_K / f_π
* ***semileptonic $K_{\ell 3}$ decays***: V_{us} from lattice determinations of $f_+(q^2 = 0)$
* test of the ***unitarity of the first-row*** of the CKM matrix using lattice inputs at the permille level

novelties

* new powerful strategy to calculate ***weak decay rates*** on the lattice including QED

  ====> feasibility demonstrated in the case of the ***leptonic $K_{\ell 2}$ and $\pi_{\ell 2}$ decays***

* lattice calculations of ***the full momentum dependence of the semileptonic form factors***

from experiments

[see also Moulson's talk in WG1]

extraction of V_{us} / V_{ud} from leptonic $K_{\ell 2}$ and $\pi_{\ell 2}$ decays

$$\Gamma\left(PS^+ \to \ell^+ \nu\right) = \frac{G_F^2}{8\pi}\left|V_{q_1 q_2}\right|^2 m_\ell^2 \left(1 - \frac{m_\ell^2}{M_{PS^+}^2}\right)^2 f_{PS^+}^2 \, M_{PS^+} \, S_{EW}\left(1 + \delta_{EM}^{PS^+}\right)$$

* S_{EW} = universal short-distance EW correction $\left(\simeq 1.0232\right)$

* EM corrections, $\delta_{EM}^{PS^+}$, estimated through ChPT with LECs parameterizing structure-dependent hadronic contributions

* relevant hadronic quantity: f_{PS^+} including strong SU(2) breaking $\left(\mathrm{m}_u \neq m_d\right)$

ChPT with LECS motivated by large-Nc methods:

$$\delta_{EM}^{K^+} - \delta_{EM}^{\pi^+} = -0.0069\,(17)$$

[see, e.g., Cirigliano and Neufeld '11]

$$\frac{\Gamma\left(K^+ \to \ell^+ \nu_\ell\right)}{\Gamma\left(\pi^+ \to \ell^+ \nu_\ell\right)} \Rightarrow \frac{\left|V_{us}\right|}{\left|V_{ud}\right|}\frac{f_{K^+}}{f_{\pi^+}} = 0.2760\,(4) \quad [0.14\ \%]$$

[Moulson '14] adopted by PDG '16 & FLAG '16

extraction of V_{us} from semileptonic $K_{\ell 3}$ decays

$$\Gamma\left(K^{+,0} \to \pi^{0,-}\ell^+\nu\right) = \frac{G_F^2 M_{K^{+,0}}^5}{192\pi^3} C_{K^{+,0}}^2 \left|V_{us}\, f_+^{K^0\pi^-}(0)\right|^2 I_{K\ell}^{(0)}\, S_{EW}\left(1+\delta_{EM}^{K^{+,0}\ell}+\delta_{SU(2)}^{K^{+,0}\pi}\right)$$

* $C_{K^{+,0}}$ = Clebsh-Gordan coefficient $\left(C_{K^+}=1/\sqrt{2}\,, C_{K^0}=1\right)$, S_{ew} = short-distance EW correction
* $I_{K\ell}^{(0)}$ = phase-space integral sensitive to the momentum dependence of vector (and scalar) form factor
* EM corrections, $\delta_{EM}^{K\ell}$, and strong SU(2) breaking, $\delta_{SU(2)}^{K\pi}$, both estimated through ChPT
* relevant hadronic quantity: vector form factor at zero 4-momentum transfer $f_+(0) \equiv f_+^{K^0\pi^-}\left(q^2=0\right)$

[Cirigliano et al. '08]

Mode	$\delta_{\rm EM}^{K\ell}$ (%)
K_{e3}^0	0.495 ± 0.110
$K_{e3}^\pm$	0.050 ± 0.125
$K_{\mu3}^0$	0.700 ± 0.110
$K_{\mu3}^\pm$	0.008 ± 0.125

$\delta_{SU(2)}^{K^0\pi^-}=0$

$\delta_{SU(2)}^{K^+\pi^0}=(2.9\pm0.4)\%$

[see FlaviaNet '10]

large local corrections (up to 10%) for Dalitz plots

$\downarrow$

momentum dependence needed for evaluating EM corrections

nice consistency between the two channels:

$$\Gamma\left(K^{+,0}\to\pi^{0,-}\ell\nu_\ell\right) \quad\Rightarrow\quad \left|V_{us}\right| f_+(0) = 0.2165\,(4) \quad [0.18\ \%]$$

[Moulson '14] adopted by FLAG '16

extraction of V_{us} from semi-inclusive τ decays

see Maltman's and Banerjeei's talks in WG1

Flavor Lattice Averaging Group 3rd review [arXiv:1607.00299]

FLAG Working Group

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colour coding

- Chiral extrapolation:
  ★ $M_{\pi,\mathrm{min}} < 200$ MeV
  ○ $200 \text{ MeV} \le M_{\pi,\mathrm{min}} \le 400$ MeV
  ■ $400 \text{ MeV} < M_{\pi,\mathrm{min}}$

- Continuum extrapolation:
  ★ at least 3 lattice spacings and at least 2 points below 0.1 fm and a range of lattice spacings satisfying $[a_{\mathrm{max}}/a_{\mathrm{min}}]^2 \ge 2$
  ○ at least 2 lattice spacings and at least 1 point below 0.1 fm and a range of lattice spacings satisfying $[a_{\mathrm{max}}/a_{\mathrm{min}}]^2 \ge 1.4$
  ■ otherwise

- Finite-volume effects:
  ★ $[M_{\pi,\mathrm{min}}/M_{\pi,\mathrm{fid}}]^2 \exp\{4 - M_{\pi,\mathrm{min}}[L(M_{\pi,\mathrm{min}})]_{\mathrm{max}}\} < 1$, or at least 3 volumes
  ○ $[M_{\pi,\mathrm{min}}/M_{\pi,\mathrm{fid}}]^2 \exp\{3 - M_{\pi,\mathrm{min}}[L(M_{\pi,\mathrm{min}})]_{\mathrm{max}}\} < 1$, or at least 2 volumes
  ■ otherwise

- Publication status:
  A published or plain update of published results
  P preprint
  C conference contribution

FLAG 1st (2011)
FLAG 2nd (2014)

FLAG 3rd updated at Oct 30th, 2016
http://itpwiki.unibe.ch/flag/index.php

$$M_{\pi,fid} = 200\ MeV$$

only results with A and no red tags enter the FLAG averages

Collaboration	Ref.	N_f	publication status	chiral extrapolation	continuum extrapolation	finite-volume errors	f_K/f_π	$f_{K^\pm}/f_{\pi^\pm}$
ETM 14E	[27]	2+1+1	A	○	★	○	1.188(11)(11)	1.184(12)(11)
FNAL/MILC 14A	[14]	2+1+1	A	★	★	★		$1.1956(10)(^{+26}_{-18})$
ETM 13F	[230]	2+1+1	C	○	★	○	1.193(13)(10)	1.183(14)(10)
HPQCD 13A	[26]	2+1+1	A	★	○	★	1.1948(15)(18)	1.1916(15)(16)
MILC 13A	[231]	2+1+1	A	★	★	★		1.1947(26)(37)
MILC 11	[232]	2+1+1	C	○	○	○		$1.1872(42)^{\dagger}_{\mathrm{stat.}}$
ETM 10E	[233]	2+1+1	C	○	○	○	$1.224(13)_{\mathrm{stat}}$	
BMW 16ulb	[234, 235]	2+1	P	★	★	★	1.182(10)(26)	
RBC/UKQCD 14B	[10]	2+1	A	★	★	★	1.1945(45)	
RBC/UKQCD 12	[31]	2+1	A	★	○	★	1.199(12)(14)	
Laiho 11	[44]	2+1	C	○	★	○		$1.202(11)(9)(2)(5)^{\dagger\dagger}$
MILC 10	[29]	2+1	C	○	★	★		$1.197(2)(^{+3}_{-7})$
JLQCD/TWQCD 10	[236]	2+1	C	○	■	★	1.230(19)	
RBC/UKQCD 10A	[144]	2+1	A	○	○	★	1.204(7)(25)	
PACS-CS 09	[94]	2+1	A	★	■	■	1.333(72)	
BMW 10	[30]	2+1	A	★	★	★	1.192(7)(6)	
JLQCD/TWQCD 09A	[237]	2+1	C	○	■	■	$1.210(12)_{\mathrm{stat}}$	
MILC 09A	[6]	2+1	C	○	★	★		$1.198(2)(^{+6}_{-8})$
MILC 09	[89]	2+1	A	○	★	★		$1.197(3)(^{+6}_{-13})$
Aubin 08	[238]	2+1	C	○	○	○		1.191(16)(17)
PACS-CS 08, 08A	[93, 239]	2+1	A	★	■	■	1.189(20)	
RBC/UKQCD 08	[145]	2+1	A	○	■	★	1.205(18)(62)	
HPQCD/UKQCD 07	[28]	2+1	A	○	○	○	1.189(2)(7)	
NPLQCD 06	[240]	2+1	A	○	■	■	$1.218(2)(^{+11}_{-24})$	
MILC 04	[107]	2+1	A	○	○	○		1.210(4)(13)
ETM 14D	[160]	2	C	★	■	○	$1.203(5)_{\mathrm{stat}}$	
ALPHA 13A	[241]	2	C	★	★	★	1.1874(57)(30)	
BGR 11	[242]	2	A	○	■	■	1.215(41)	
ETM 10D	[215]	2	C	○	★	○	$1.190(8)_{\mathrm{stat}}$	
ETM 09	[32]	2	A	○	★	○	1.210(6)(15)(9)	
QCDSF/UKQCD 07	[243]	2	C	○	○	★	1.21(3)	

† Result with statistical error only from polynomial interpolation to the physical point.
†† This work is the continuation of Aubin 08.

Table 14: Colour code for the data on the ratio of decay constants: f_K/f_π is the pure QCD $SU(2)$-symmetric ratio, while $f_{K^\pm}/f_{\pi^\pm}$ is in pure QCD including the $SU(2)$ isospin-breaking correction.

$$\boxed{\frac{f_{K^\pm}}{f_{\pi^\pm}} = \frac{f_K}{f_\pi}\sqrt{1+\delta_{SU(2)}}}$$

three methods to include strong SU(2)-breaking corrections

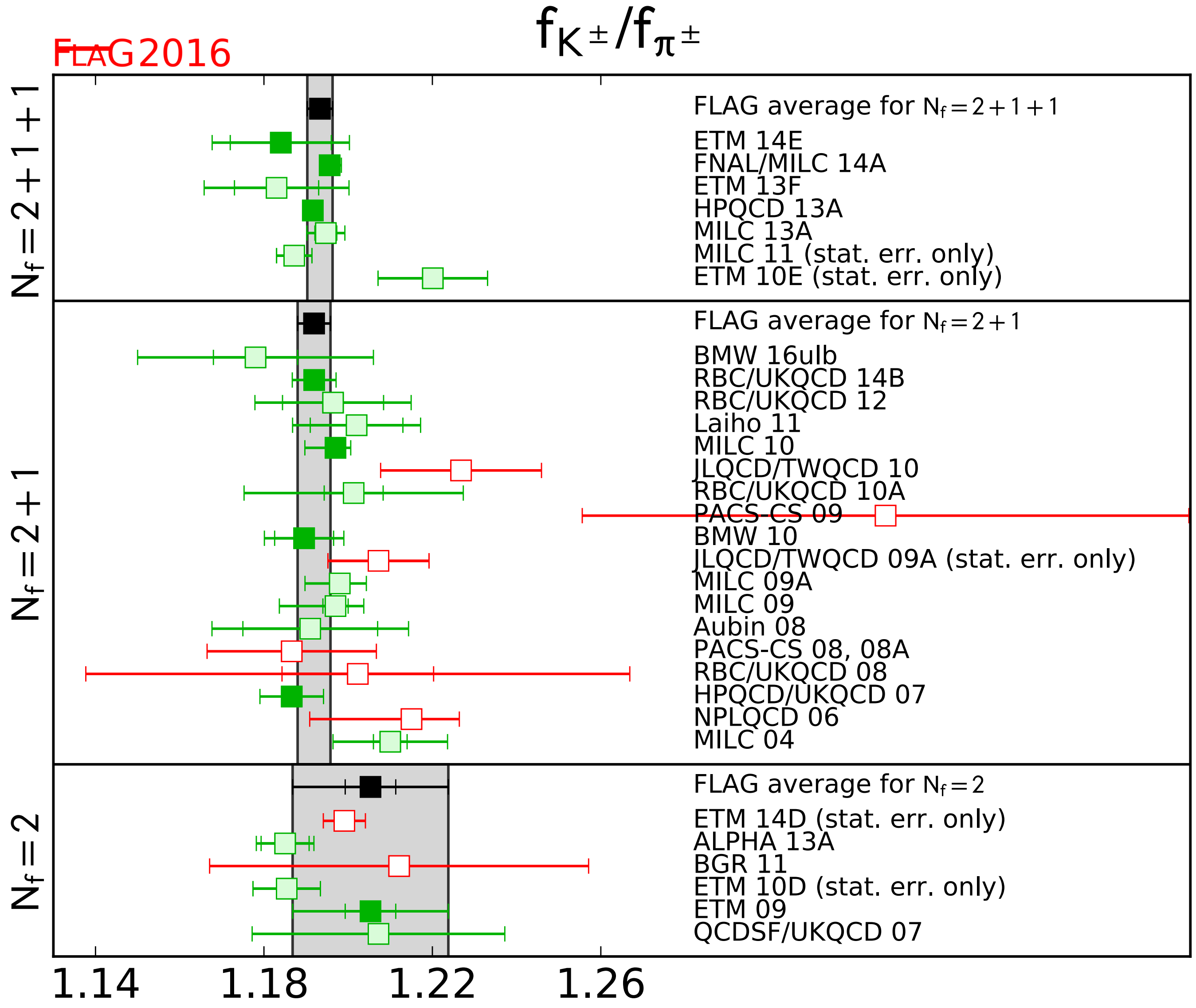
- extrapolation up to m_u or m_d
- insertion of the scalar density
- estimate using ChPT

$$\delta_{SU(2)}^{ChPT} = \frac{m_d - m_u}{m_s - m_{ud}}\left[1 - \frac{f_K}{f_\pi} + \frac{1}{32\pi^2 f_0^2}\cdot \left(M_K^2 - M_\pi^2 - M_\pi^2 \log\frac{M_K^2}{M_\pi^2}\right)\right] + \ldots$$

$\delta_{SU(2)}^{(extrapolation)} = -0.0054\,(14)$ HPQCD

$\delta_{SU(2)}^{(insertion)} = -0.0080\,(4)$ RM123

$\delta_{SU(2)}^{(ChPT)} = -0.0043\,(11)$ C&N '11



$f_{K^\pm}/f_{\pi^\pm} = 1.193\,(3)$ $\quad N_f = 2+1+1$ $\quad$ HPQCD, FNAL/MILC, ETMC

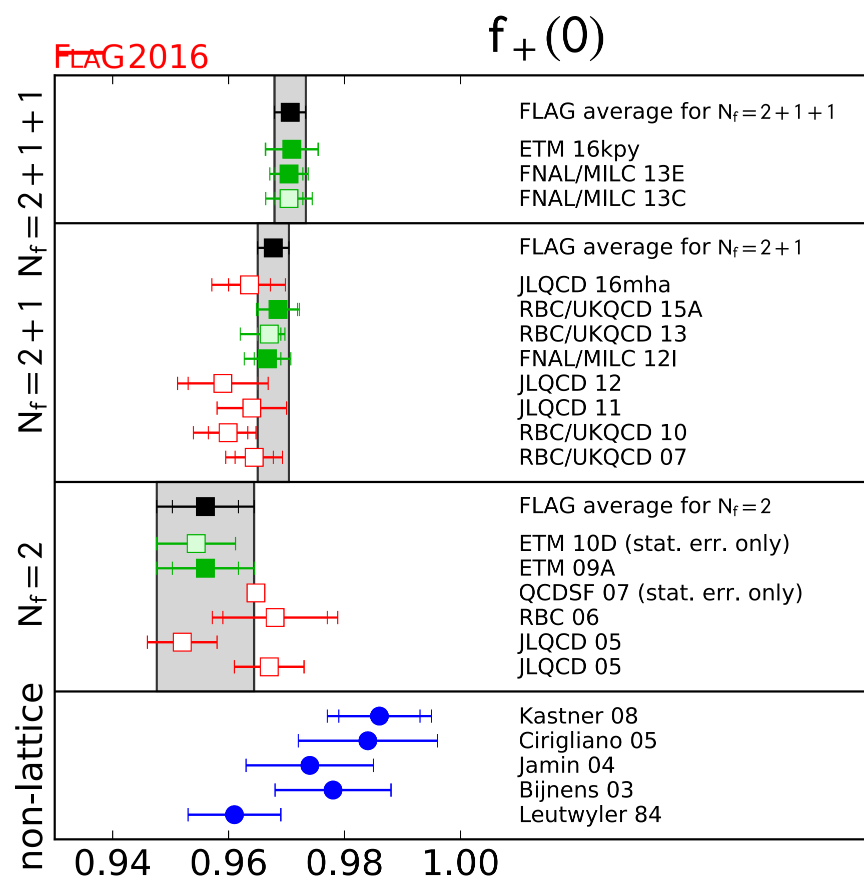
$f_{K^\pm}/f_{\pi^\pm} = 1.192\,(4)$ $\quad N_f = 2+1$ $\quad$ HPQCD/UKQCD, MILC, BMW, RBC/UKQCD

$f_{K^\pm}/f_{\pi^\pm} = 1.205\,(18)$ $\quad N_f = 2$ $\quad$ ETMC

precision at the level of ~ 0.25 - 0.4% on f_{K^+}/f_{π^+} both for $N_f = 2+1$ and $N_f = 2+1+1$

Collaboration	Ref.	N_f	publication status	chiral extrapolation	continuum extrapolation	finite-volume errors	$f_+(0)$
ETM 16kpy	[207]	2+1+1	A	○	★	○	$0.9709(45)(9)$
FNAL/MILC 13E	[22]	2+1+1	A	★	★	★	$0.9704(24)(22)$
FNAL/MILC 13C	[208]	2+1+1	C	★	★	★	$0.9704(24)(32)$
JLQCD 16mha	[209]	2+1	C	○	■	★	$0.9636(36)(^{+50}_{-54})$
RBC/UKQCD 15A	[24]	2+1	A	★	○	○	$0.9685(34)(14)$
RBC/UKQCD 13	[210]	2+1	A	★	○	○	$0.9670(20)(^{+18}_{-46})$
FNAL/MILC 12I	[23]	2+1	A	○	○	★	$0.9667(23)(33)$
JLQCD 12	[211]	2+1	C	○	■	★	$0.959(6)(5)$
JLQCD 11	[212]	2+1	C	○	■	★	$0.964(6)$
RBC/UKQCD 10	[213]	2+1	A	○	■	★	$0.9599(34)(^{+31}_{-47})(14)$
RBC/UKQCD 07	[214]	2+1	A	○	■	★	$0.9644(33)(34)(14)$
ETM 10D	[215]	2	C	○	★	○	$0.9544(68)_{stat}$
ETM 09A	[25]	2	A	○	○	○	$0.9560(57)(62)$
QCDSF 07	[216]	2	C	■	■	★	$0.9647(15)_{stat}$
RBC 06	[217]	2	A	■	■	★	$0.968(9)(6)$
JLQCD 05	[218]	2	C	■	■	★	$0.967(6)$, $0.952(6)$

Table 13: Colour code for the data on $f_+(0)$.



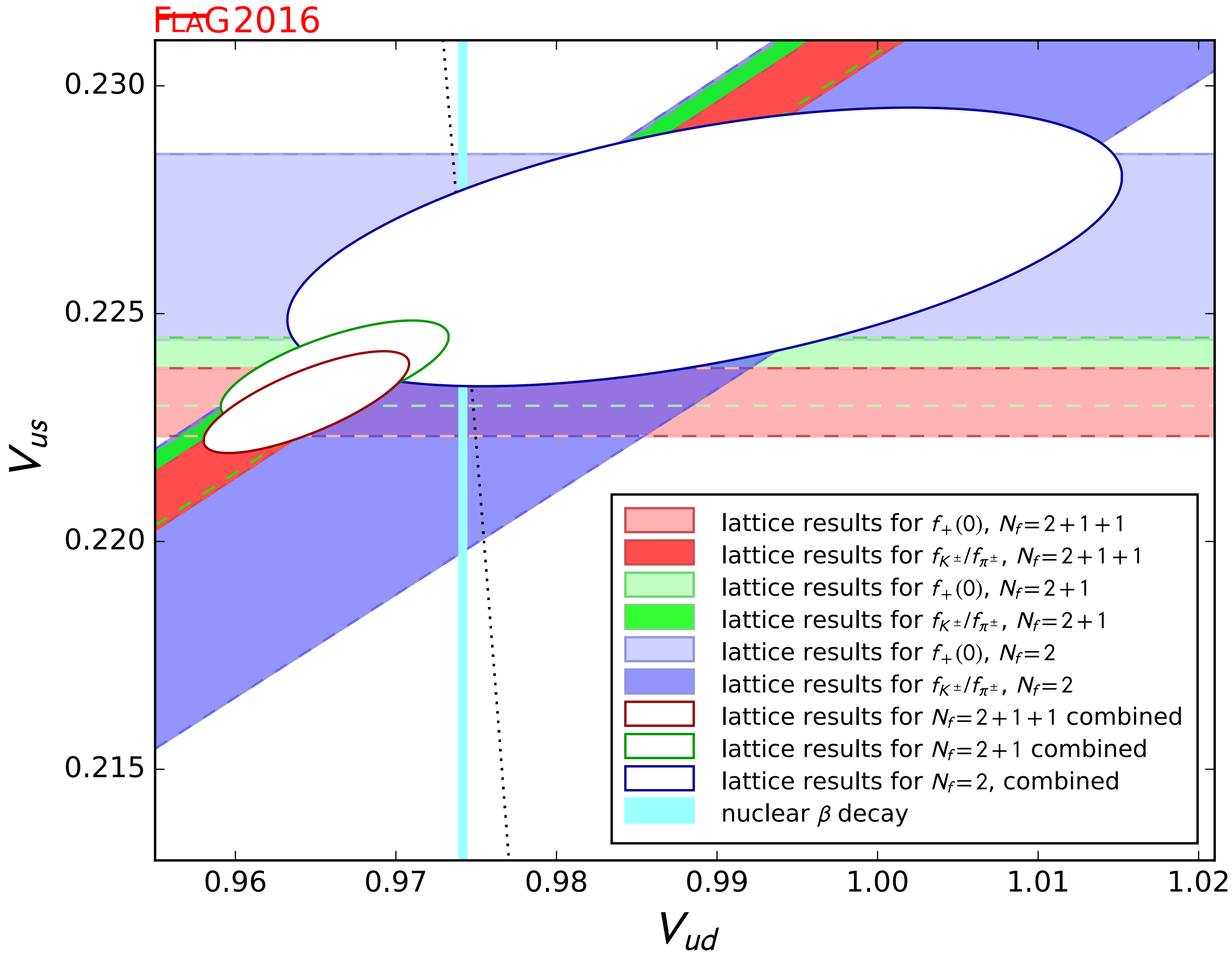
$$f_+(0) = 0.9706\,(27) \qquad N_f = 2+1+1 \qquad \text{FNAL/MILC, ETMC}$$

$$f_+(0) = 0.9677\,(27) \qquad N_f = 2+1 \qquad \text{FNAL/MILC, RBC/UKQCD}$$

$$f_+(0) = 0.9560\,(84) \qquad N_f = 2 \qquad \text{ETMC}$$

precision at the level of $\sim 0.3\%$ on $f_+(0)$ both for $N_f = 2+1$ and $N_f = 2+1+1$

unitarity of the CKM first-row



experimental results **[Moulson '14]**

from $K_{\ell 2}$ decays: $\left|\frac{V_{us}}{V_{ud}}\right|\frac{f_{K^+}}{f_{\pi^+}} = 0.2760\,(4)$

from $K_{\ell 3}$ decays: $|V_{us}|\,f_+(0) = 0.2165\,(4)$

$$|V_u|^2 \equiv |V_{ud}|^2 + |V_{us}|^2 + \cancel{|V_{ub}|^2}$$

$$\text{SM:} \quad |V_u|^2 \equiv 1$$

- adopting $N_f = 2+1+1$ lattice results: $\quad |V_u|^2 = 0.980\,(9) \qquad \approx 2.2\sigma$

- using $|V_{ud}| = 0.97417\,(21)$ from superallowed nuclear β decays **[H&T '15]**

$$|V_u|^2 = 0.9988\,(6) \qquad \text{from } f_+(0) \qquad \approx 2\sigma$$

$$|V_u|^2 = 0.9998\,(5) \qquad \text{from } f_{K^+}/f_{\pi^+} \qquad \approx 0.4\sigma$$

* current precision has reached the level of **few permille** on both f_{K^+}/f_{π^+} and $f_+(0)$

* improvements can be expected from the production of new gauge ensembles
  - with better statistics
  - closer to the physical point
  - at finer lattice spacing
  - at larger lattice volume

precision at the permille level (or even below) is foreseeable in the next future, **but ...**

* **EM correction** for $K_{\ell 2}$ decays: $\sqrt{1+\delta_{EM}^{K^+}-\delta_{EM}^{\pi^+}} = 0.9966(8)$ **[ChPT]**

uncertainty at the **permille level** (with some model-dependence)

ambitious goal: evaluation of weak decay rates on the lattice including QCD and QED

* recently QED has been included in lattice QCD simulations in the case of the hadron spectrum

RM123 '13, BMW '15, ...

* **however** => for the spectrum no IR divergencies
  => for decay rates IR divergencies can be cancelled by summing up virtual and real photons

* it's not enough to add the EM interaction to the quark action, but **new strategies** should be developed in order to evaluate decay rates on the lattice

such a new strategy has been recently proposed **[PRD91 (2015) 074506]**

and applied to the leptonic decays of kaons and pions **[arXiv: 1610.09668 (LAT '16)]**

the new procedure is based on a double expansion at LO in α_{em} and $\delta m = m_d - m_u$

PRD87 (2013) 114505

PRD91 (2015) 074506

1) the emission of virtual photons at leading order in the EM coupling is evaluated on the lattice

2) the subtraction of the infrared divergence is computed for a point-like meson using the finite lattice volume as the infrared regulator

3) the emission of virtual+real photons from a point-like meson is added using a photon mass for the infrared regularization

$$\Gamma = \left[\Gamma_0^{lattice}(L) - \Gamma_0^{pt}(L)\right] + \left[\Gamma_0^{pt}(m_\gamma) + \Gamma_1^{pt}(m_\gamma)\right]$$

master formula for the leptonic decay rate

$$\Gamma\left(PS \to \ell\nu[\gamma]\right) = \Gamma^{(tree)}(PS \to \ell\nu) \cdot \left[1 + \delta R_{PS}(\Delta E_\gamma)\right]$$

tree level: $\Gamma^{(tree)}(PS \to \ell\nu) = \frac{G_F^2}{8\pi}|V_{q_1q_2}|^2 m_\ell^2 \left(1 - \frac{m_\ell^2}{M_{PS}^2}\right)^2 \left[f_{PS}^{(0)}\right]^2 M_{PS}$

$$f_{PS}^{(0)} \equiv \frac{p_{PS}^\mu}{M_{PS}^2}\langle 0|\bar{q}_2\gamma_\mu\gamma_5 q_1|PS\rangle$$

$$M_{PS} = M_{PS}^{(0)} + \alpha_{em}\delta_{EM}M_{PS} + (m_d - m_u)\delta_{SU(2)}M_{PS}$$

$$\delta R_{PS}(\Delta E_\gamma) = \frac{2}{\pi}\log\left(\frac{M_Z}{M_W}\right) + 2\delta\left[\frac{A_{PS}}{f_{PS}^{(0)}M_{PS}}\right] + \delta\Gamma^{pt}(\Delta E_\gamma)$$

short-distance EW correction not included in G_F (μ lifetime)

virtual photon emissions calculated on the lattice (using the lattice volume as IR regulator)

EM correction (virtual + real photons up to energy ΔE_γ) for a point-like PS meson (using a photon mass as IR regulator)

* δA_{PS} and $\delta\Gamma^{pt}(\Delta E_\gamma)$ are separately IR finite and independent on the specific IR regularization

calculation of $\delta\Gamma^{pt}(\Delta E_\gamma)$

$$\delta\Gamma^{pt}\left(\Delta E_\gamma\right) = \delta\Gamma_0^{pt} + \delta\Gamma_1^{pt}\left(\Delta E_\gamma\right)$$

the sum is IR finite (Bloch-Nordsieck mechanism)

FIG. 8. One loop diagrams contributing to the wave-function renormalization of a pointlike pion.

* <u>virtual photons $\delta\Gamma_0^{pt}$</u>

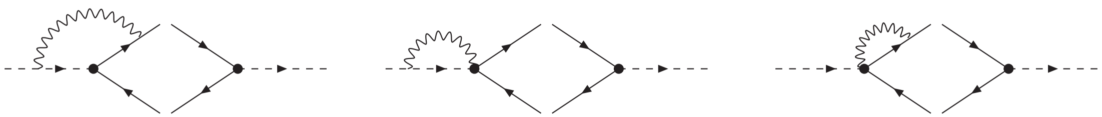


FIG. 9. Radiative corrections to the pion-lepton vertex. The diagrams represent $O(\alpha)$ contributions to $\Gamma_0^{\rm pt}$. The left part of each diagram represents a contribution to the amplitude and the right part the tree-level contribution to the Hermitian conjugate of the amplitude. The corresponding diagrams containing the radiative correction on the right-hand side of each diagram are also included.

* <u>real photons $\delta\Gamma_1^{pt}(\Delta E_\gamma)$</u>

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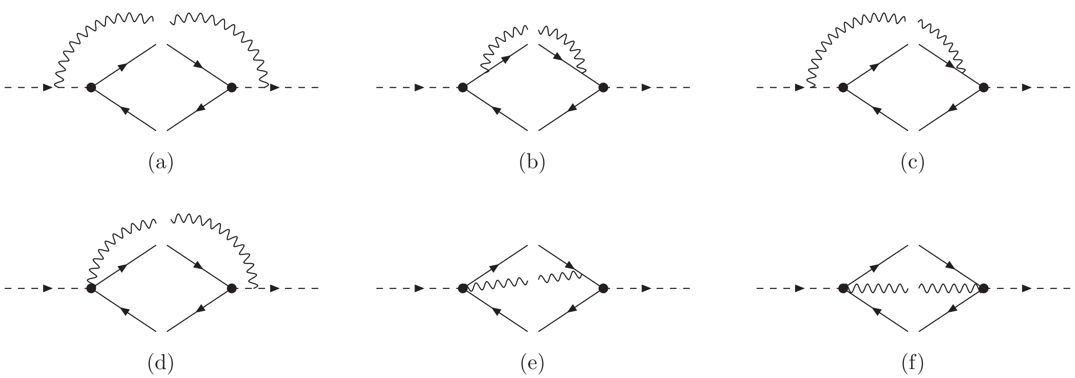

FIG. 10. Diagrams contributing to $\Gamma_1(\Delta E)$. For diagrams (c), (d) and (e) the "conjugate" contributions in which the photon vertices on the left and right of each diagram are interchanged are also to be included. The labels (a)–(f) are introduced to identify the individual diagrams when describing their evaluation in the text.

[PRD91 (2015) 074506]

$$\begin{aligned}
\delta\Gamma^{pt}(\Delta E_\gamma) \;=\; & \frac{1}{4\pi}\Big\{3\log(M_{PS}^2/M_W^2) - 3 + \log(r_\ell^2) \\
& - 4\log(r_E^2) + \frac{2-10r_\ell^2}{1-r_\ell^2}\log(r_\ell^2) \\
& - 2\frac{1+r_\ell^2}{1-r_\ell^2}\log(r_\ell^2)\log(r_E^2) \\
& - 4\frac{1+r_\ell^2}{1-r_\ell^2}\mathrm{Li}_2(1-r_\ell^2) \\
& + \frac{3+r_E^2-6r_\ell^2-4r_E(1-r_\ell^2)}{(1-r_\ell^2)^2}\log(1-r_E) \\
& + r_E\frac{4-r_E-4r_\ell^2}{(1-r_\ell^2)^2}\log(r_\ell^2) \\
& - r_E\frac{28r_\ell^2+3r_E-22}{2(1-r_\ell^2)^2} \\
& - 4\frac{1+r_\ell^2}{1-r_\ell^2}\mathrm{Li}_2(r_E)\Big\}
\end{aligned}$$

$$r_\ell = m_\ell / M_{PS}\,,\quad r_E = 2\Delta E_\gamma / M_{PS}$$

$\Delta E_\gamma \sim$ 10-20 MeV
for the point-like assumption to be valid

calculation of δA_{PS}

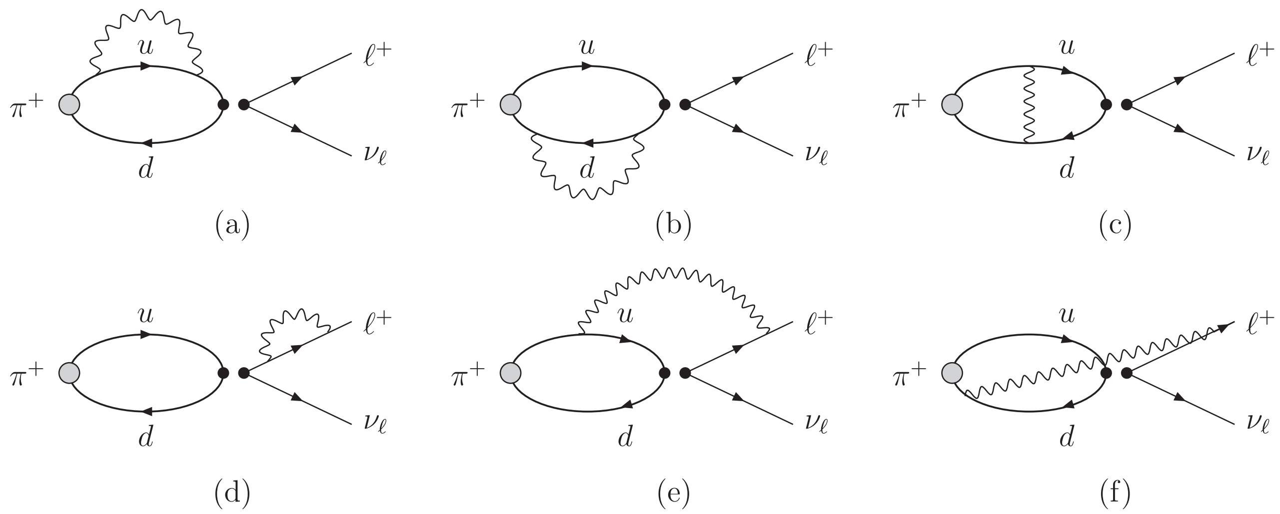

FIG. 5. Connected diagrams contributing at $O(\alpha)$ to the amplitude for the decay $\pi^+ \to \ell^+\nu_l$. The labels (a)–(f) are introduced to identify the individual diagrams when describing their evaluation in the text.

virtual photons between quarks and/or lepton

connected diagrams

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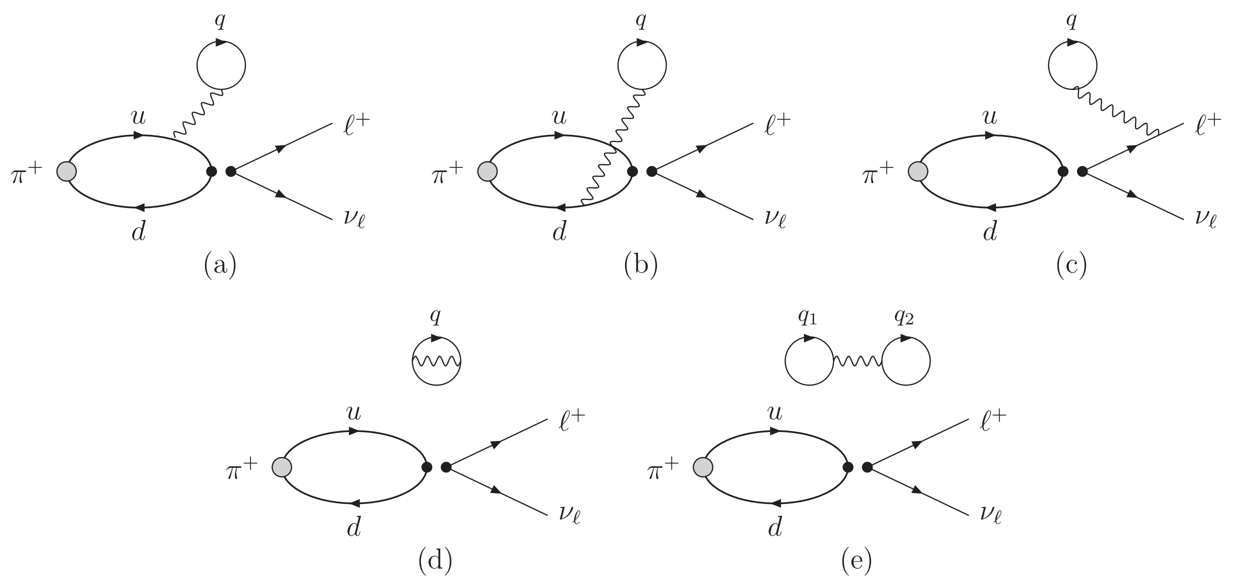
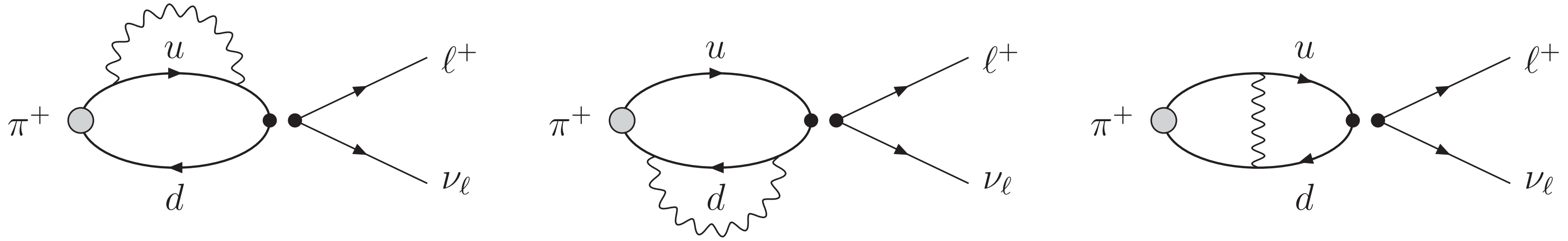


FIG. 6. Disconnected diagrams contributing at $O(\alpha)$ to the amplitude for the decay $\pi^+ \to \ell^+\nu_l$. The curly line represents the photon, and a sum over quark flavors q, q_1 and q_2 is to be performed. The labels (a)–(e) are introduced to identify the individual diagrams when describing their evaluation in the text.

disconnected diagrams

quenched QED: $e_f^{sea} = 0$

* virtual photons between quarks: lattice calculation



(a) (b) (c)

$$\delta C^{(qq)}(t) = -\frac{1}{2}\sum_{\vec{x},x_1,x_2} \langle 0|T\left\{J^\rho_{ew}(0)\, j^{em}_\mu(x_1)\, j^{em}_\mu(x_2)\, \phi^\dagger_{PS}(\vec{x},-t)\right\}|0\rangle\, \Delta_{em}(x_1,x_2)\,\frac{p^\rho_{PS}}{M_{PS}}$$

(V-A) quark current

conserved em quark current

PS interpolating field

photon propagator

tree level: $C_0(t) = \sum_{\vec{x}} \langle 0|T\left\{J^\rho_{ew}(0)\,\phi^\dagger_{PS}(\vec{x},-t)\right\}|0\rangle \frac{p^\rho_{PS}}{M_{PS}}$

large time distances: $C_0(t) + \delta C^{(qq)}(t) \xrightarrow[t>>a]{} \frac{Z_{PS}A^{(qq)}_{PS}}{2M_{PS}}\left[e^{-M_{PS}t} - e^{-M_{PS}(T-t)}\right]$

$$M_{PS} = M^{(0)}_{PS} + \delta M_{PS}, \quad A^{(qq)}_{PS} = A^{(0)}_{PS} + \delta A^{(qq)}_{PS}, \quad Z_{PS} = Z^{(0)}_{PS} + \delta Z_{PS}$$

$$\frac{\delta C^{(qq)}(t)}{C_0(t)} \xrightarrow[t>>a]{} \frac{\delta\left[Z_{PS}A^{(qq)}_{PS}\right]}{Z^{(0)}_{PS}A^{(0)}_{PS}} + \frac{\delta M_{PS}}{M^{(0)}_{PS}} f(t) \qquad f(t) \equiv M^{(0)}_{PS}\left(\frac{T}{2} - t\right)\frac{e^{-M^{(0)}_{PS}t} + e^{-M^{(0)}_{PS}(T-t)}}{e^{-M^{(0)}_{PS}t} - e^{-M^{(0)}_{PS}(T-t)}} - 1 \approx -M^{(0)}_{PS}t$$

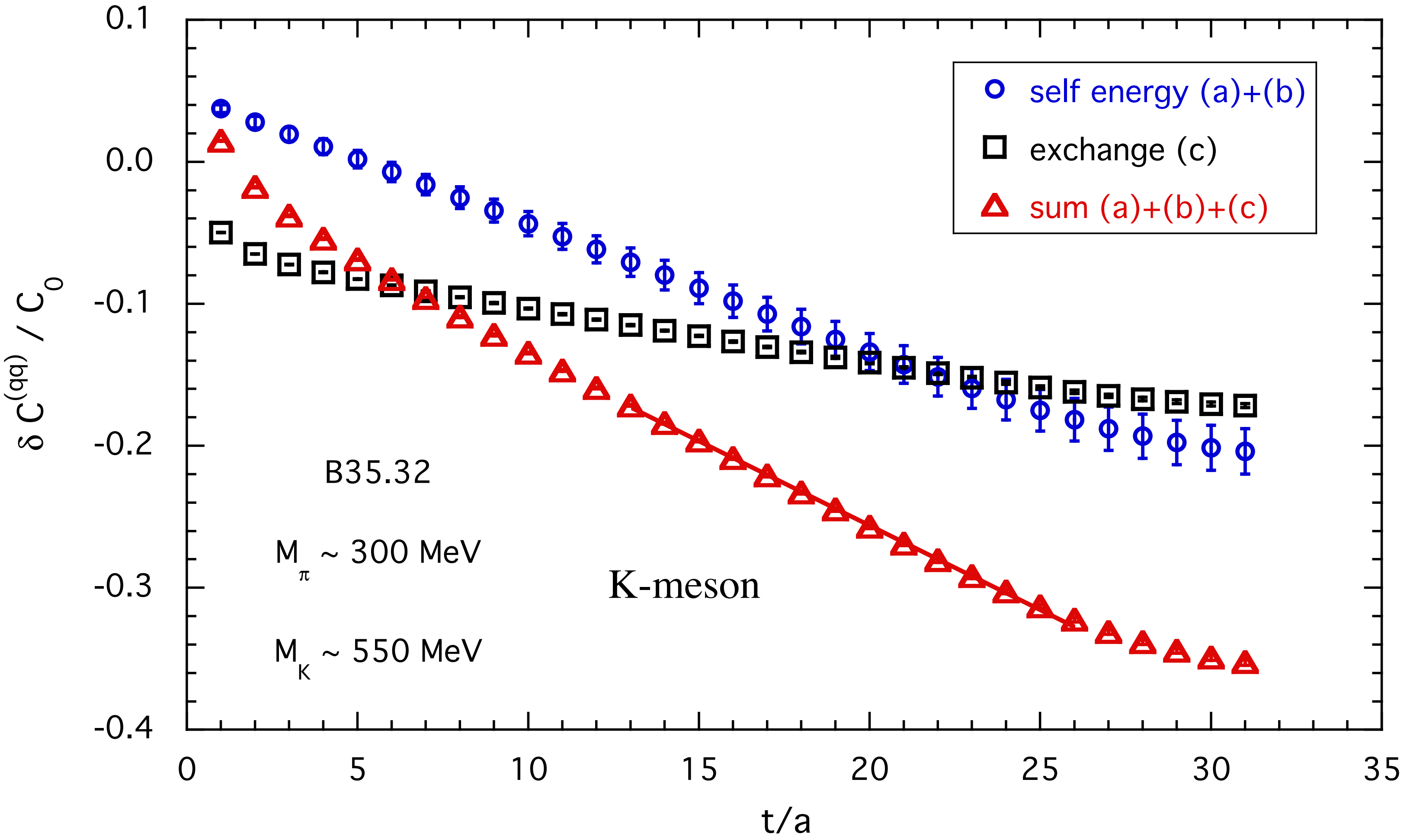
***** δM_{PS} from the slope and $\delta\left[Z_{PS}A^{(qq)}_{PS}\right]$ from the intercept *****

gauge ensembles from the European Twisted Mass Collaboration (ETMC)

* $N_f = 2+1+1$ dynamical sea quarks: two light mass-degenerate flavors, strange and charm sea quarks with masses close to their physical value

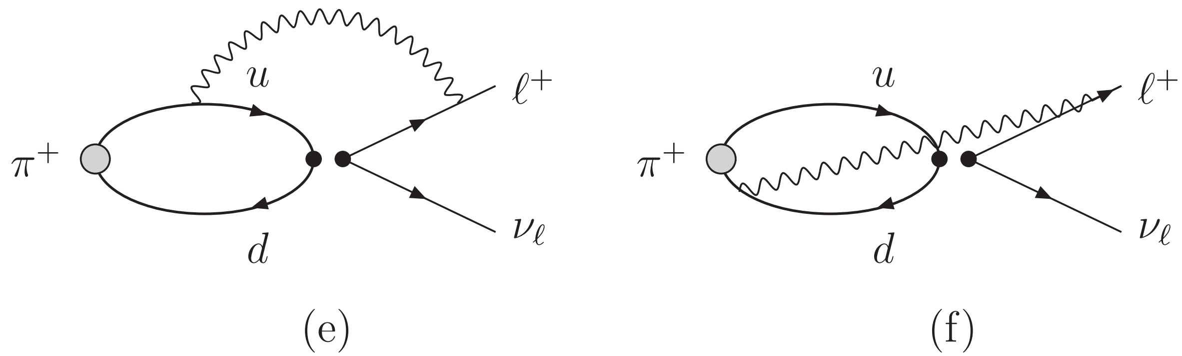
* three values of the lattice spacing: 0.0885, 0.0815, 0.0619 fm

* pion masses simulated in the range between 220 and 470 MeV



δM_{PS} from the slope $\delta\left[Z_{PS} A_{PS}^{(qq)}\right]$ from the intercept

* virtual photons between quarks and final lepton: lattice calculation



[times the tree-level leptonic part]

$$\delta C^{(q\ell)}(t) = -\sum_{\vec{x},x_1,x_2} \langle 0| T\left\{ J^\rho_{ew}(0) j^{em}_\mu(x_1) \phi^\dagger_{PS}(\vec{x},-t)\right\}|0\rangle \Delta_{em}(x_1,x_2) e^{E_\ell t_2 - i\vec{p}_\ell\cdot\vec{x}_2}$$

$$\cdot\, \bar{u}(p_\nu)\gamma_\rho(1-\gamma_5) S^\ell(0,x_2)\gamma_\mu v(p_\ell)\left[\bar{v}(p_\ell)\gamma_\sigma(1-\gamma_5)u(p_\nu)\frac{p^\sigma_{PS}}{M_{PS}}\right]$$

$S^\ell(0,x) =$ free twisted-mass lepton propagator $\quad E_\ell = \sqrt{m_\ell^2 + \vec{p}_\ell^{\,2}}, \quad E_\ell + E_\nu = M^{(0)}_{PS} \quad \vec{p}_\ell$ injected via non-periodic b.c.

tree-level: $C^{(q\ell)}_0(t) = C_0(t) Tr(p_\ell, p_{PS})$

leptonic trace: $Tr(p_\ell, p_{PS}) = \bar{u}(p_\nu)\gamma_\rho(1-\gamma_5)v(p_\ell)\bar{v}(p_\ell)\gamma_\sigma(1-\gamma_5)u(p_\nu)\frac{p^\rho_{PS}}{M_{PS}}\frac{p^\sigma_{PS}}{M_{PS}}$

* expanding the (V-A) structure of the quark EW current:

$$\delta C^{(q\ell)}(t) = Z_A\left[\delta C^{(V_0)}(t) + \delta C^{(V_k)}(t)\right] + Z_V\left[\delta C^{(A_0)}(t) + \delta C^{(A_k)}(t)\right]$$

(twisted-mass renormalization)

$$\delta C^{(q\ell)}(t) \xrightarrow[t>>a]{} \frac{Z^{(0)}_{PS}}{2M^{(0)}_{PS}} \delta A^{(q\ell)}_{PS}\, Tr(p_\ell, p_{PS})\left[e^{-M^{(0)}_{PS}t} \pm \text{ backward signals}\right]$$

depending on the time/spatial components

subtraction of IR divergence and of universal FSEs

$$\frac{\delta A_{PS}}{A_{PS}^{(0)}} \to \frac{\delta A_{PS}(L)}{A_{PS}^{(0)}} - \frac{\delta A^{(pt)}(L)}{A_{PS}^{(0)}}$$

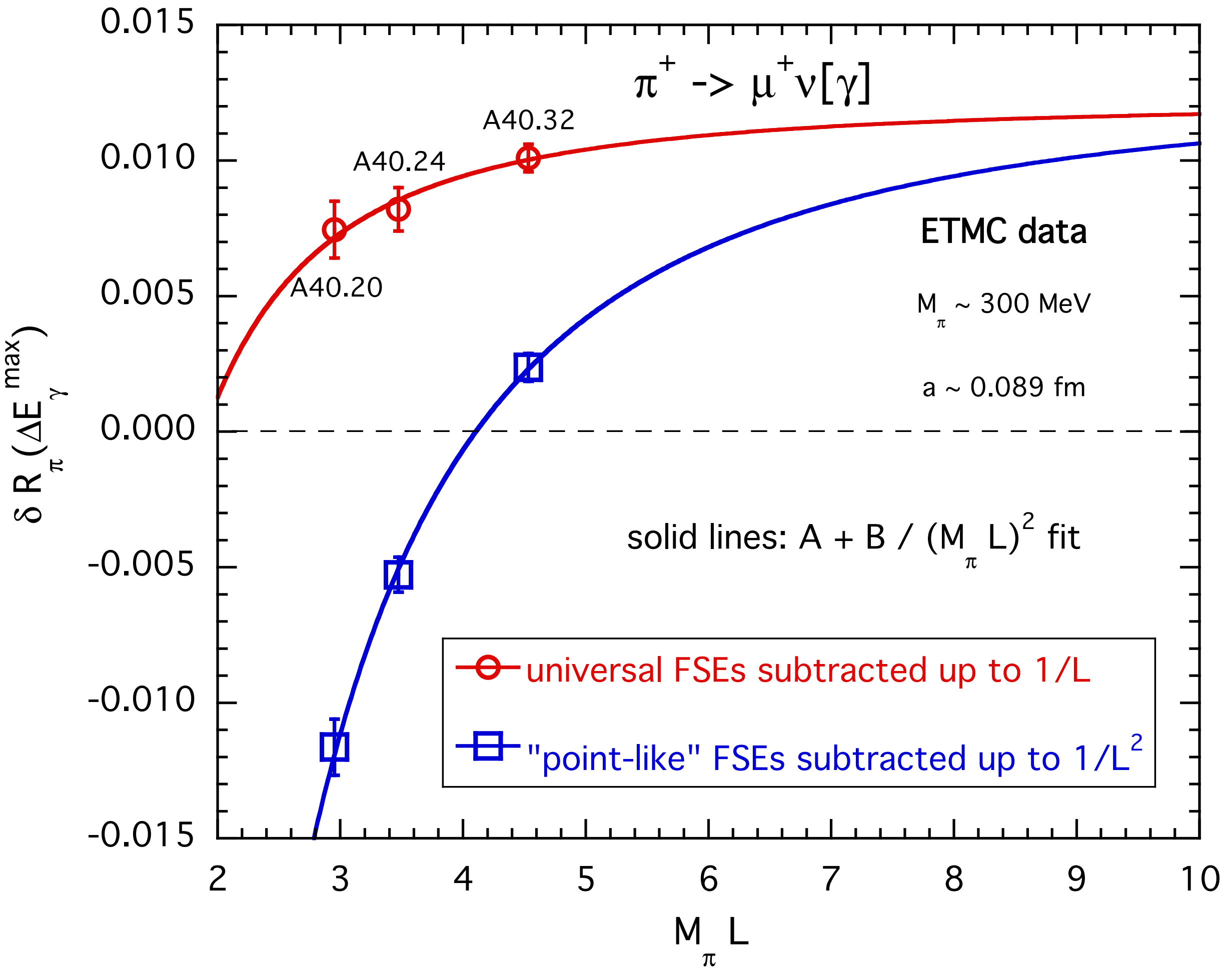
virtual photon emission from a point-like meson using the lattice volume as IR regulator

arXiv:1610.09668 : $$\frac{\delta A^{(pt)}(L)}{A_{PS}^{(0)}} = b_{IR}\log(M_{PS}L) + b_0 + \frac{b_1}{M_{PS}L} + \frac{b_2 + b_2^{SD}}{(M_{PS}L)^2} + \frac{b_3 + b_3^{SD}}{(M_{PS}L)^3}$$

$b_i = b_i(r_\ell, \vec{p}_\ell)$ [known]

$r_\ell = m_\ell / M_{PS}$

* (unknown) structure-dependent FSEs start at order $(1/L)^2$ ⟹ compare $\begin{cases} \text{up to } 1/L \text{ subtraction: } b_2 = b_3 = 0 \\ \text{up to } 1/L^2 \text{ subtraction: } b_3 = 0 \end{cases}$



$$\delta R_\pi(\Delta E_\gamma^{\max}) = \frac{2}{\pi}\log\left(\frac{M_Z}{M_W}\right) + 2\frac{\delta A_\pi^{qq} + \delta A_\pi^{q\ell}}{A_\pi^{(0)}} - 2\frac{\delta A^{(pt)}(L)}{A_\pi^{(0)}} - 2\frac{\delta M_\pi}{M_\pi^{(0)}} + \alpha_{em}(Z_1 + Z_2) + \delta\Gamma^{pt}(\Delta E_\gamma^{\max})$$

$$\Delta E_\gamma^{\max} \cong 29.6 \text{ MeV}$$

residual FSE still visible

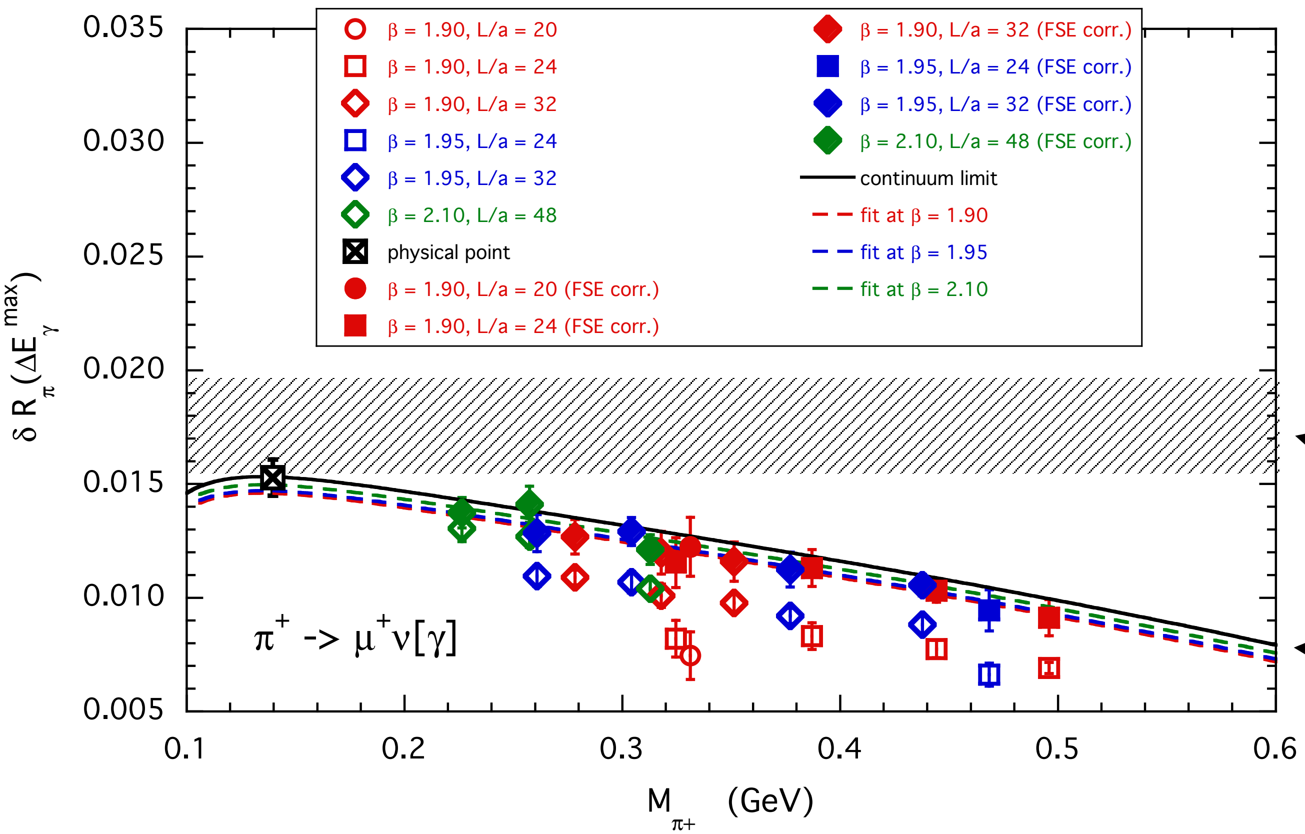
$$b_2 + b_2^{SD} << b_2^{SD}$$

* chiral extrapolation [Knecht et al., EPJC 12 (2000) 469]

$$\delta R_\pi\left(\Delta E_\gamma^{\max}\right) = 4\pi E(\mu) + \frac{3}{4\pi}\log\left(\frac{\xi}{\mu^2}\right) + A_1\xi + Da^2 + \delta\Gamma^{pt}\left(\Delta E_\gamma^{\max}\right) + K_\pi^{FSE}(L) \qquad \xi \equiv \frac{M_\pi^2}{(4\pi f_0)^2}$$

residual (structure-dependent) FSEs: $K_\pi^{FSE}(L) = \frac{K_2}{(M_\pi L)^2} + \frac{K_2^\ell}{(E_\ell L)^2}$ $\qquad E, A_1, D, K_2, K_2^\ell$: 5 free parameters

ETMC data



open markers: ETMC data with subtraction of universal FSEs up to 1/L

full markers: ETMC data with subtraction of both universal and structure-dependent FSEs

$\delta R_\pi^{ChPT}\left(\Delta E_\gamma^{\max}\right) = 0.0176\,(21)$

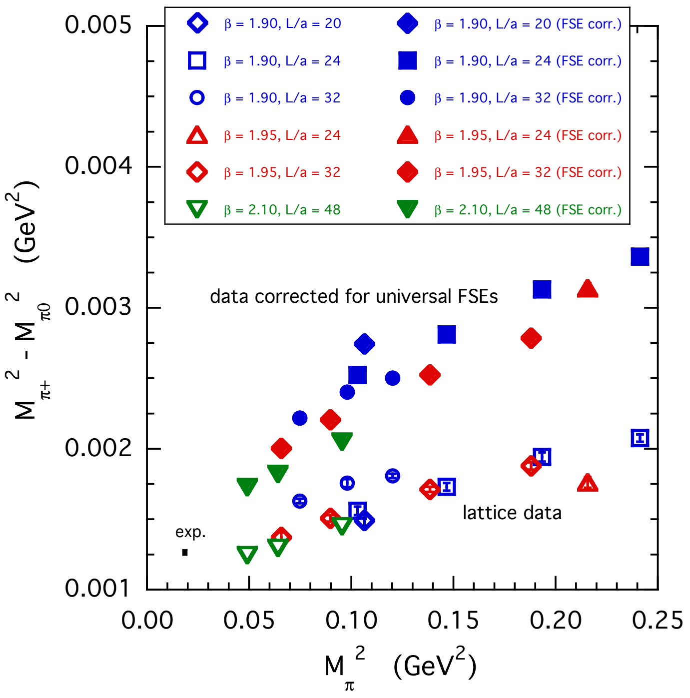
fit including the chiral log $\quad \chi^2_{dof} \simeq 0.7$

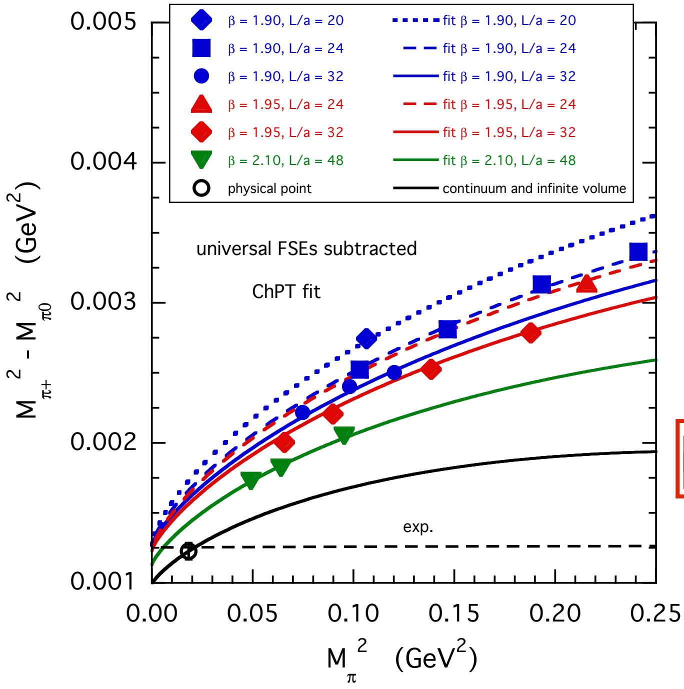
* adopting different fitting functions (chiral vs. polynomial) with different FSE subtractions, one has

$$\begin{aligned}\delta R_\pi^{phys}\left(\Delta E_\gamma^{\max}\right) &= 0.0169\,(8)_{stat+fit}\;(11)_{chiral}\;(7)_{FSE}\;(2)_{a^2}\\ &= 0.0169\,(8)_{stat+fit}\;(13)_{syst} = 0.0169\,(15)\end{aligned}$$

$$\frac{\delta R_\pi^{phys}\left(\Delta E_\gamma^{\max}\right)}{\delta R_\pi^{ChPT}\left(\Delta E_\gamma^{\max}\right)} = 0.9993\,(26)$$

results for charged/neutral kaon and pion masses





pion

universal FSE: $-\alpha_{em}\kappa\left(2+M_\pi L\right)/L^2$

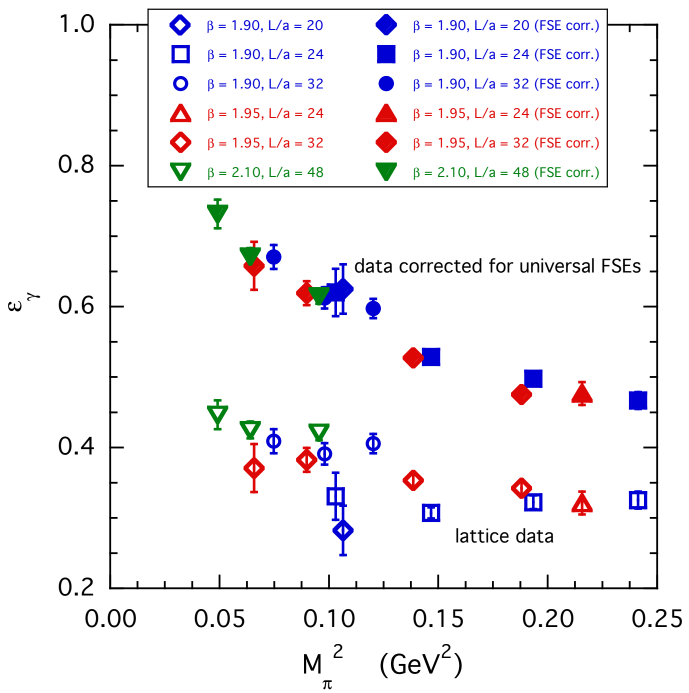
Hayakawa&Uno **[PTP '08]**

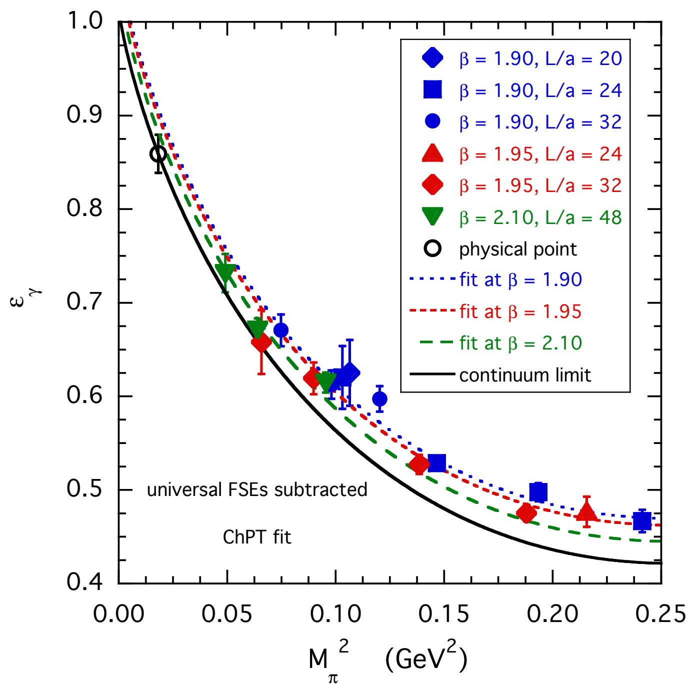
BMW: QED$_L$ on T^4 **[Science '15]**

residual FSEs still visible

$$\left[M_{\pi^+}^2 - M_{\pi^0}^2\right]^{phys} = 1.226\,(58)_{stat}\,(96)_{syst}\; 10^{-3}\ \text{GeV}^2$$

$$\left[M_{\pi^+}^2 - M_{\pi^0}^2\right]^{exp} = 1.2612\,(1)\,10^{-3}\ \text{GeV}^2$$





kaon

$$\varepsilon_\gamma = \frac{\left[M_{K^+}^2 - M_{K^0}^2 + M_{\pi^0}^2 - M_{\pi^+}^2\right]^{QED}}{M_{\pi^+}^2 - M_{\pi^0}^2}$$

$$\varepsilon_\gamma^{phys} = 0.833\,(22)_{stat}\,(28)_{syst} \qquad \overline{MS}(2\ \text{GeV})$$

$$\varepsilon_\gamma^{FLAG} = 0.7\,(3)$$ **[arXiv:1607.00299]**

$$\delta_{SU(2)} M_K = -4.66\,(6)_{stat}\,(22)_{syst}$$

$$m_d - m_u = 2.69\,(5)_{stat}\,(13)_{syst}\ \text{MeV}$$

* **K/π ratio:** $\delta R_{K\pi}\left(\Delta E_\gamma\right) = \delta R_K\left(\Delta E_\gamma\right) - \delta R_\pi\left(\Delta E_\gamma\right)$

$K^+ \to \mu^+\nu[\gamma]: \Delta E_\gamma^{max} \cong 235.5$ MeV

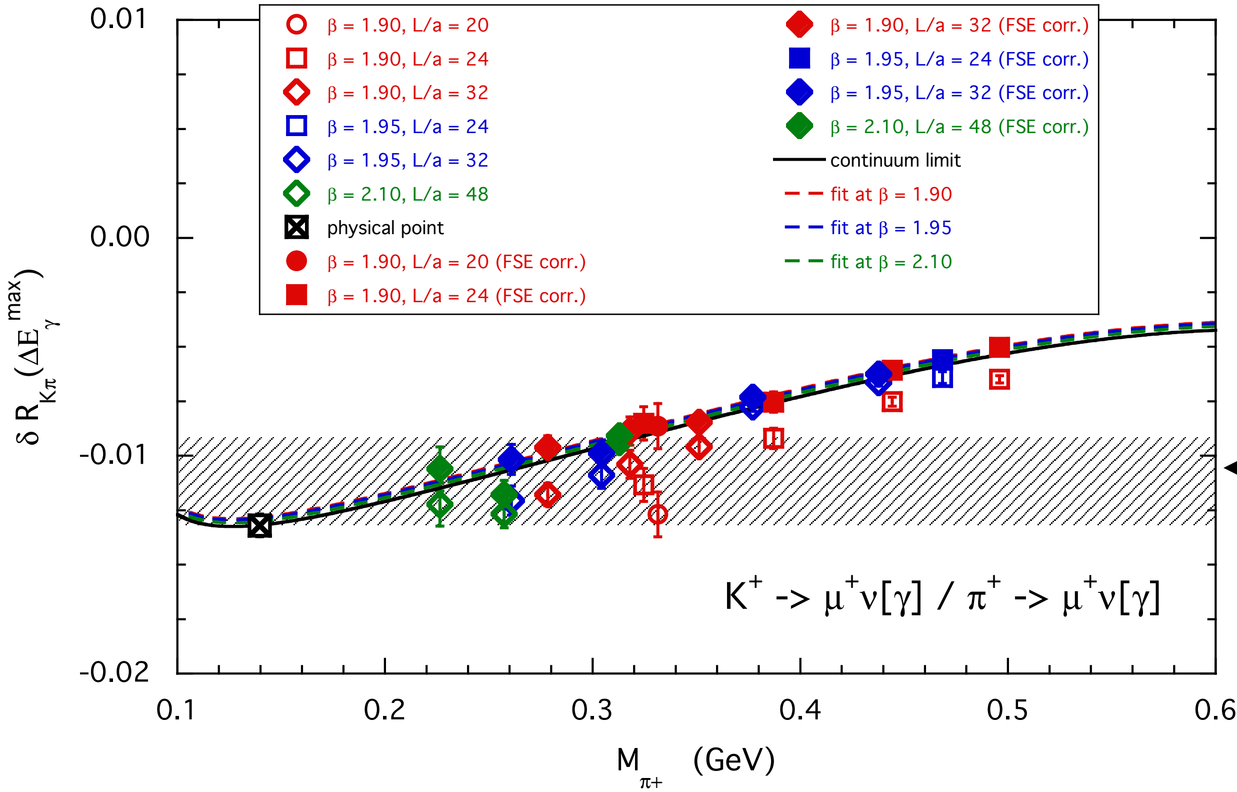
$\pi^+ \to \mu^+\nu[\gamma]: \Delta E_\gamma^{max} \cong 29.6$ MeV

$$\delta R_{K\pi}\left(\Delta E_\gamma^{max}\right) = \tilde{A}_0 - \frac{3}{4\pi}\log\left(\frac{M_\pi^2}{M_K^2}\right) + \tilde{A}_1\xi + \tilde{A}_2\xi^2 + \tilde{D}a^2 + \delta\Gamma_K^{pt}\left(\Delta E_\gamma^{max}\right) - \delta\Gamma_\pi^{pt}\left(\Delta E_\gamma^{max}\right) + K_{K\pi}^{FSE}(L)$$

$$K_{K\pi}^{FSE}(L) = \frac{\tilde{K}_2}{(M_K L)^2} + \frac{\tilde{K}_2^\ell}{\left(E_\ell^{(K)}L\right)^2} - \frac{K_2}{(M_\pi L)^2} - \frac{K_2^\ell}{\left(E_\ell^{(\pi)}L\right)^2}$$

$\tilde{A}_0, \tilde{A}_1, \tilde{A}_2, \tilde{D}, \tilde{K}_2, \tilde{K}_2^\ell$: 6 free parameters

ETMC data



open markers: ETMC data with subtraction of universal FSEs up to 1/L

full markers: ETMC data with subtraction of both universal and structure-dependent FSEs

fit including the chiral log $\chi^2_{dof} \simeq 1.1$

$\delta R_{K\pi}^{ChPT}\left(\Delta E_\gamma^{max}\right) = -0.0112\,(21)$

$\delta_{EM} R_{K\pi}^{ChPT}\left(\Delta E_\gamma^{max}\right) = -0.0069\,(17)$

$\delta_{SU(2)} R_{K\pi}^{ChPT} = -0.0043\,(11)$ **[Cirigliano&Neufeld '11]**

* adopting different fitting functions (chiral vs. polynomial) with different FSE subtractions, one has

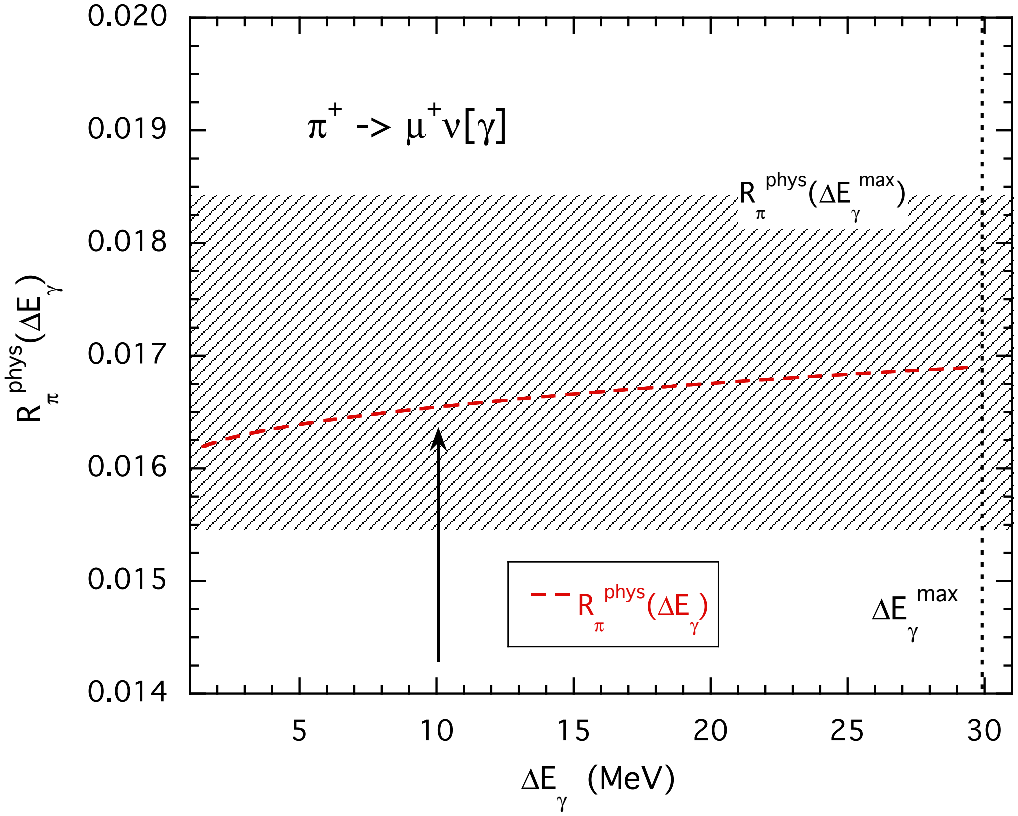
$$\boxed{\begin{aligned}\delta R_{K\pi}^{phys}\left(\Delta E_\gamma^{max}\right) &= -0.0137\,(11)_{stat+fit}\,(6)_{chiral}\,(1)_{FSE}\,(1)_{a^2} \\ &= -0.0137\,(11)_{stat+fit}\,(6)_{syst} = -0.0137\,(13)\end{aligned}}$$

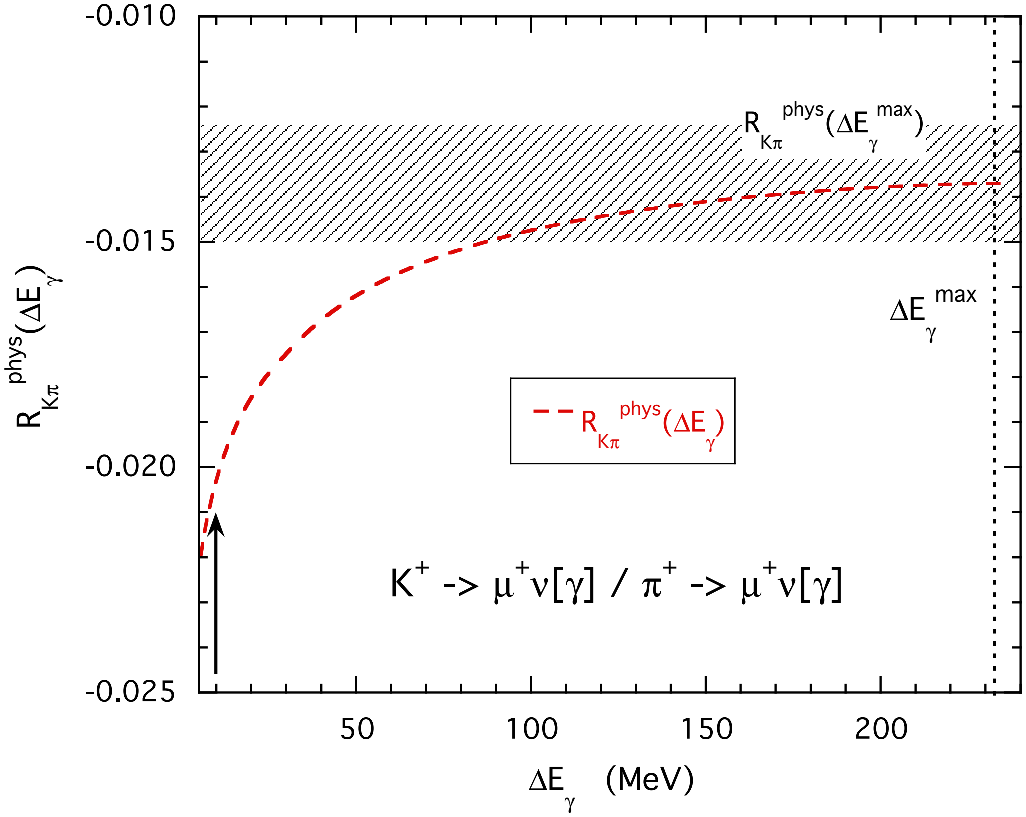
$$\boxed{\frac{\delta R_{K\pi}^{phys}\left(\Delta E_\gamma^{max}\right)}{\delta R_{K\pi}^{ChPT}\left(\Delta E_\gamma^{max}\right)} = 1.22\,(26)}$$

two open issues

* **removal of the qQED approximation** ⇨ **evaluation of (fermionic) disconnected diagrams**

* **maximum photon energy: $\Delta E_\gamma \sim$ 10-20 MeV for the point-like assumption to be valid**





~ OK for the pion case ($\Delta E_\gamma^{max} \sim$ 30 MeV) NOT OK for the kaon case ($\Delta E_\gamma^{max} \sim$ 235 MeV)

cuts in the photon energy for experimental data should be (re)considered

semileptonic $K_{\ell 3}$ decays

$$\Gamma\left(K^{+,0} \to \pi^{0,-}\ell^+\nu\right) = \frac{G_F^2 M_{K^{+,0}}^5}{192\pi^3} C_{K^{+,0}}^2 \left|V_{us}\, f_+^{K^0\pi^-}(0)\right|^2 I_{K\ell}^{(0)}\, S_{EW} \left(1 + \delta_{EM}^{K^{+,0}\ell} + \delta_{SU(2)}^{K^{+,0}\pi}\right)$$

* IB correction $\left(1+\delta_{EM}^{K\ell} + \delta_{SU(2)}^{K\pi}\right)$: not yet available from lattice,

but previous approach can be extended to $K_{\ell 3}$ decays (**work is in progress ...**)

* $f_+(0) \equiv f_+^{K^0\pi^-}\left(q^2 = 0\right)$ is till now the only relevant hadronic quantity

- dedicated studies at $q^2 = 0$ to avoid the systematic due to the momentum extrapolation (RBC/UKQCD coll.)
- however such a systematics is largely **sub-leading** (see arXiv:1602.04113) and the EM corrections requires **the knowledge of the momentum dependence of the ff's**

* the phase-space integral $I_{K\ell}^{(0)}$, depending on $f_{+,0}\left(q^2\right)/f_+(0)$, is evaluated using the experimental data

- experimental kinematical range: $m_\ell^2 \leq q^2 \leq \left(M_K - M_\pi\right)^2 \simeq 0.129\ \text{GeV}^2$
- Taylor expansion: $f_{+,0}\left(q^2\right)/f_+(0) = 1 + \lambda'_{+,0} \frac{q^2}{M_\pi^2} + \frac{1}{2}\lambda''_{+,0}\left(\frac{q^2}{M_\pi^2}\right)^2 + \frac{1}{6}\lambda'''_{+,0}\left(\frac{q^2}{M_\pi^2}\right)^3 + \ldots$

* **strong correlations** among slopes, $\lambda'_{+,0}$, and curvatures, $\lambda''_{+,0}$, and ...

* dispersive approach [Bernard et al. '09]

$$f_+^{disp}(q^2)/f_+(0) = e^{\frac{q^2}{M_\pi^2}\left[\Lambda_+ + H(q^2)\right]}, \qquad f_0^{disp}(q^2)/f_+(0) = e^{\frac{q^2}{q_{CT}^2}\left[\log(C) - G(q^2)\right]}$$

$$q_{cut}^2 = (M_K + M_\pi)^2$$
$$q_{CT}^2 = M_K^2 - M_\pi^2$$

$$H(q^2) = \frac{M_\pi^2 q^2}{\pi}\int_{q_{cut}^2}^{\infty} ds \frac{\phi_+(s)}{s^2(s - q^2 - i\varepsilon)}, \qquad G(q^2) = \frac{q_{CT}^2(q_{CT}^2 - q^2)}{\pi}\int_{q_{cut}^2}^{\infty} ds \frac{\phi_0(s)}{s(s - q_{CT}^2)(s - q^2 - i\varepsilon)}$$

- in the elastic region $\phi_{+(0)}$ is the P(S)-wave phase shift of the $(K\pi)_{I=1/2}$ scattering
- both $\mathrm{H}(q^2)$ and $G(q^2)$ can be evaluated numerically, obtaining:

$$\lambda_+' = \Lambda_+$$
$$\lambda_+'' = \Lambda_+^2 + 5.79(97)\cdot 10^{-4}$$
$$\lambda_+''' = \Lambda_+^3 + 5.79(97)\cdot 10^{-4}\cdot 3\Lambda_+ + 2.99(21)\cdot 10^{-5}$$

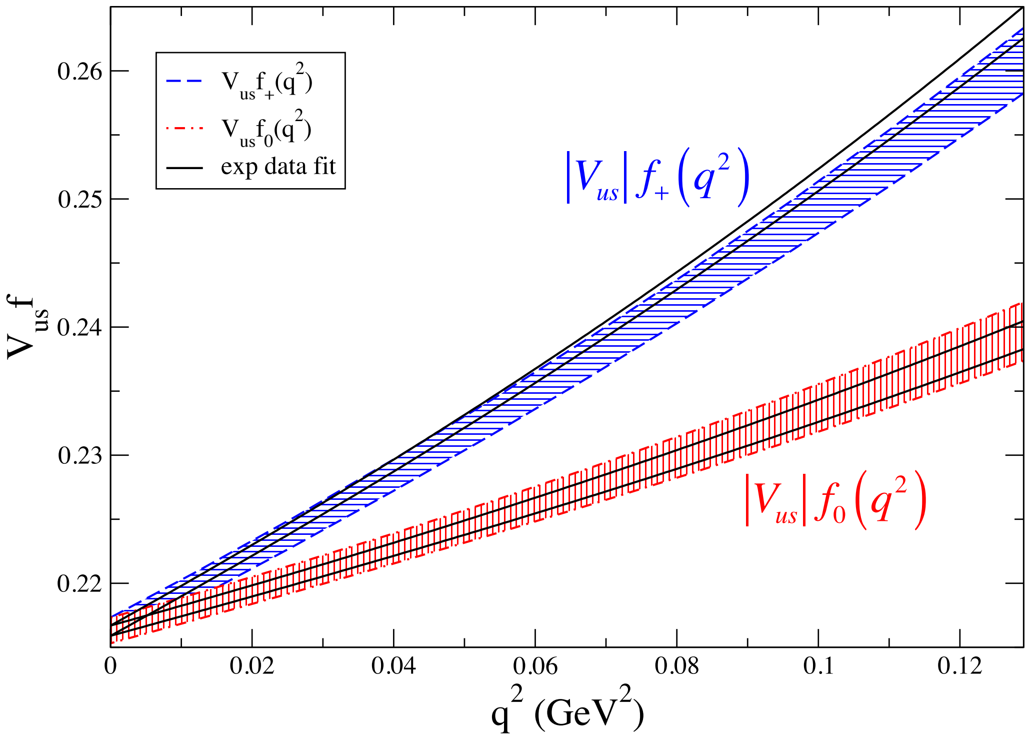
$$\lambda_0' = M_\pi^2\left[\log(C) - 0.0398(44)\right]/q_{CT}^2$$
$$\lambda_0'' = (\lambda_0')^2 + 4.16(56)\cdot 10^{-4}$$
$$\lambda_0''' = (\lambda_0')^3 + 4.16(56)\cdot 10^{-4}\cdot 3\lambda_0' + 2.72(21)\cdot 10^{-5}$$

KTeV, KLOE, NA48/2, ISTRA+:

$$\Lambda_+^{exp.} = 25.75(36)\cdot 10^{-3}$$
$$\log(C)^{exp.} = 0.1985(70)$$

[Moulson '14]

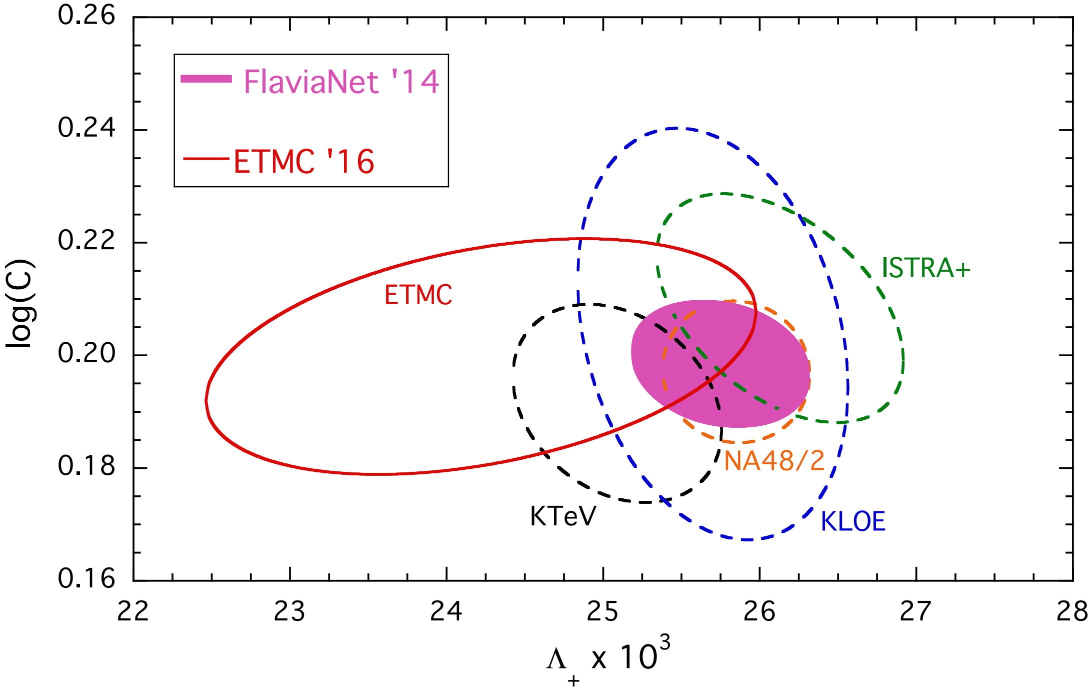
* **momentum dependence calculated only by ETM Coll.** [Carrasco et al., arXiv:1602.04113]



results at the physical point agree with experimental data

$$f_+(0) = 0.9709(46) \Rightarrow |V_{us}| = 0.2230(4)_{\text{exp.}}(11)_{f_+(0)}$$

ETMC '16	FlaviaNet '14
$\Lambda_+ = 24.22(1.16)\cdot 10^{-3}$	$\Lambda_+ = 25.75(36)\cdot 10^{-3}$
$\log(C) = 0.1998(138)$	$\log(C) = 0.1985(70)$



correlations among $f_+(0)$, Λ_+ and log(C) have been calculated

precision expected yo be improved in the next future

more lattice calculations of the momentum dependence of ff's are called for

CONCLUSIONS AND PERSPECTIVES

* lattice determinations of f_{K^+}/f_{π^+} and $f_+(0)$ have reached the precision of **few permille**

[FLAG arXiv:1607.00299 and web update]

* improvements can be expected in the next future from the production of new gauge ensembles and a precision at the permille level (or even below) is foreseeable in the future, **but ...**

uncertainties on electromagnetic and strong SU(2) corrections are at the permille level

next target: evaluation of weak decay rates on the lattice including QCD and QED

* a new strategy to calculate ***QED corrections to hadronic processes***, although very challenging, **is within the reach of present lattice technologies** [PRD91 (2015) 074506]

* the **first lattice results** on the electromagnetic effects in the leptonic decay rates **$\pi^+ \to \mu^+\nu[\gamma]$** and **$K^+ \to \mu^+\nu[\gamma]$** have been already achieved [arXiv: 1610.09668]

$$\boxed{\delta R_{\pi}^{phys}\left(\Delta E_{\gamma}^{\max}\right)/\delta R_{\pi}^{ChPT}\left(\Delta E_{\gamma}^{\max}\right)=0.9993\,(26)} \qquad \boxed{\delta R_{K\pi}^{phys}\left(\Delta E_{\gamma}^{\max}\right)/\delta R_{K\pi}^{ChPT}\left(\Delta E_{\gamma}^{\max}\right)=1.22\,(26)}$$

* extension to semileptonic **$K_{\ell 3}$ decays** is in progress

* importance of studying the **momentum dependence** of the semileptonic $K_{\ell 3}$ form factors

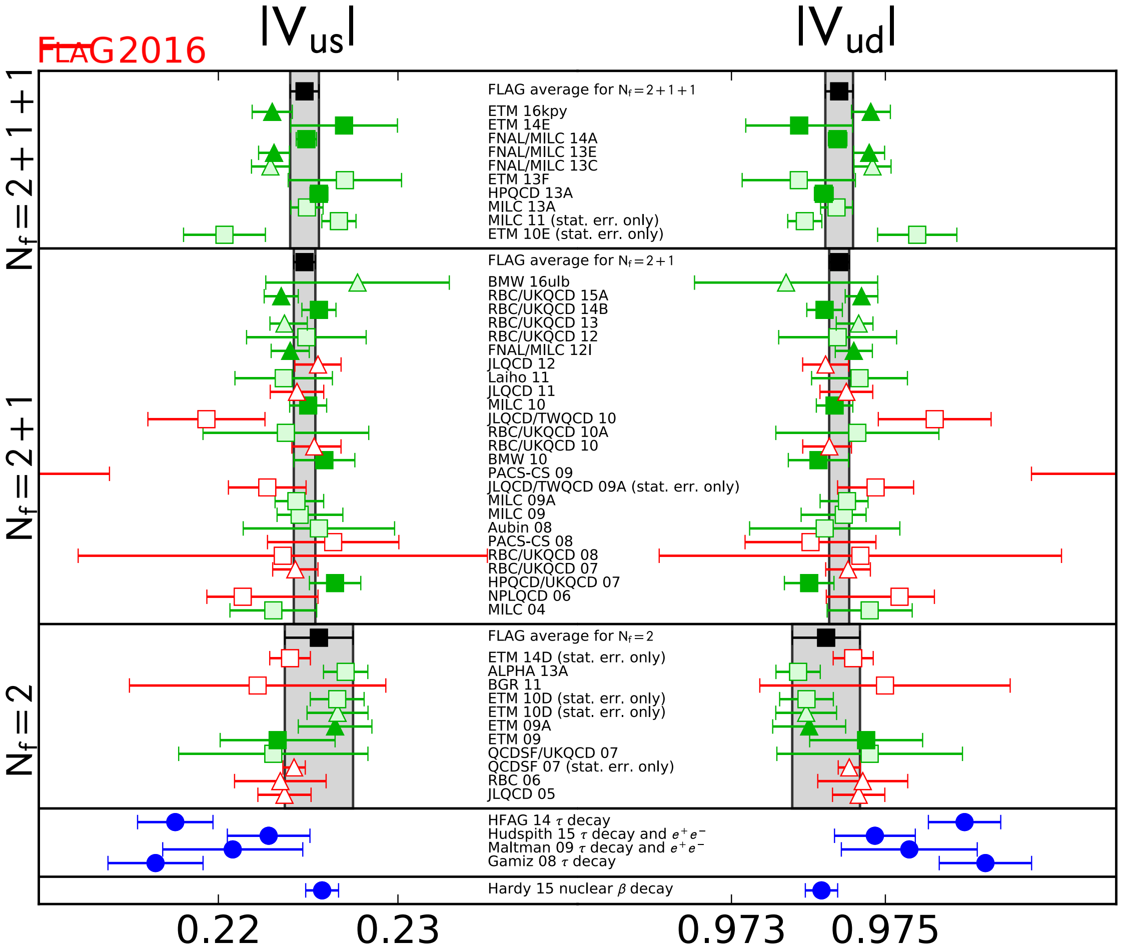
BACKUP SLIDES

analysis within the SM

from $K_{\ell 2}$ decays: $\left|\frac{V_{us}}{V_{ud}}\right| \frac{f_{K^+}}{f_{\pi^+}} = 0.2760\,(4)$

from $K_{\ell 3}$ decays: $|V_{us}|\, f_+(0) = 0.2165\,(4)$

from either $\frac{f_{K^+}}{f_{\pi^+}}$ or $f_+(0)$ one can obtain both V_{us} and V_{ud} assuming **CKM unitarity**



	Ref.	$\lvert V_{us}\rvert$	$\lvert V_{ud}\rvert$
$N_f = 2+1+1$		0.2248(8)	0.97440(18)
$N_f = 2+1$		0.2248(6)	0.97440(13)
$N_f = 2$		0.2256(19)	0.97423(44)
β decay	[185]	0.2258(9)	0.97417(21)
τ decay	[199]	0.2165(26)	0.9763(6)
τ decay + e^+e^-	[198]	0.2208(39)	0.9753(9)
τ decay + e^+e^-	[201]	0.2238(23)	0.9749(5)

new implementation of the relevant sum rules
[see Maltman's and Banerjeei's talks in WG1]

triangles: from $f_+(0)$ squares: from $\frac{f_{K^+}}{f_{\pi^+}}$

European Twisted Mass Collaboration (ETMC) setup

- $N_f = 2+1+1$ dynamical sea quarks: two light mass-degenerate flavors, strange and charm sea quarks close to the physical ones
- Wilson twisted-mass action for sea and valence up/down quarks, Osterwalder-Seiler action for valence strange (and charm) quark
- Iwasaki action for the gluons
- maximal twist guarantees an automatic O(a)-improvement for the above non-unitary setup

gauge ensembles

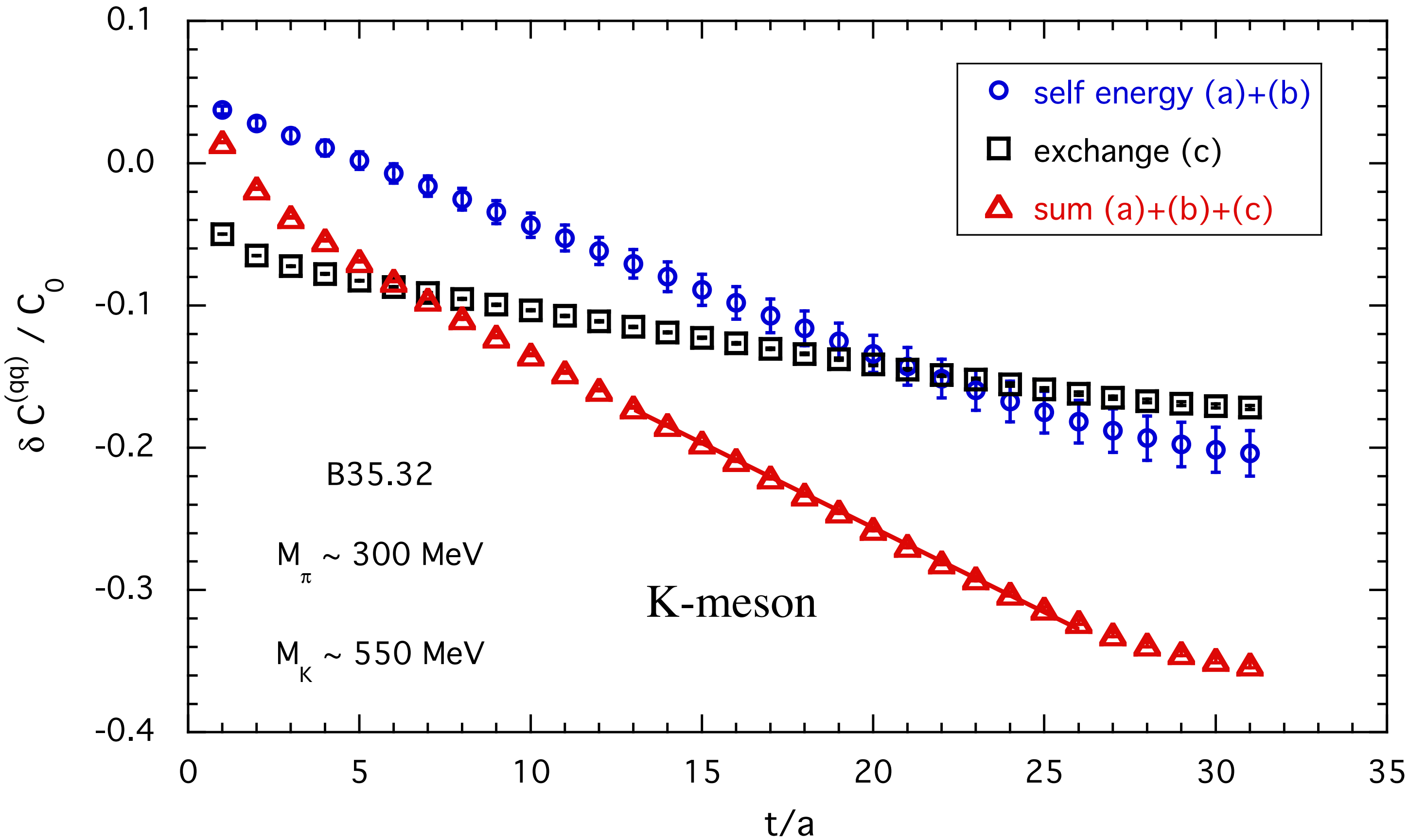
three values of the lattice spacing:
a ~ 0.0885 (36), 0.0815 (30), 0.0619 (18) fm

lattice sizes from 1.8 to 3 fm
$3 < M_\pi L < 6$

pion masses from 225 to 500 MeV

the strange quark mass at each β is calculated using the physical m_s mass and Z_m obtained by ETMC in NPB 887 (2014)

ensemble	β	V/a^4	$a\mu_{sea} = a\mu_\ell$	$a\mu_\sigma$	$a\mu_\delta$	N_{cfg}	$a\mu_s$	M_{π^+} (MeV)	M_{K^+} (MeV)	L (fm)	$M_\pi L$
$A30.32$	1.90	$32^3 \times 64$	0.0030	0.15	0.19	150	0.0236	278	564	2.9	4.0
$A40.32$			0.0040			100		318	573		4.6
$A50.32$			0.0050			150		351	581		5.1
$A40.24$		$24^3 \times 48$	0.0040			150		325	579	2.1	3.5
$A60.24$			0.0060			150		387	594		4.2
$A80.24$			0.0080			150		444	615		4.8
$A100.24$			0.0100			150		496	636		5.4
$A40.20$		$20^3 \times 48$	0.0040			150		331	583	1.8	3.0
$B25.32$	1.95	$32^3 \times 64$	0.0025	0.135	0.170	150	0.0209	261	542	2.6	3.5
$B35.32$			0.0035			150		304	551		4.1
$B55.32$			0.0055			150		377	574		5.0
$B75.32$			0.0075			80		438	596		5.8
$B85.24$		$24^3 \times 48$	0.0085			150		468	609	2.0	4.7
$D15.48$	2.10	$48^3 \times 96$	0.0015	0.12	0.1385	100	0.0161	226	526	3.0	3.4
$D20.48$			0.0020			100		257	529		3.9
$D30.48$			0.0030			100		313	546		4.8



δM_{PS} from the slope

$\delta\left[Z_{PS}A_{PS}^{(qq)}\right]$ from the intercept

* need to subtract δZ_{PS}

$$\delta C^{PS}(t) = -\frac{1}{2}\int d^3\vec{x}d^4x_1d^4x_2\,\langle 0|T\left\{\phi_{PS}(0)j_\mu^{em}(x_1)j_\mu^{em}(x_2)\phi_{PS}^\dagger(\vec{x},-t)\right\}|0\rangle\,\Delta_{em}(x_1,x_2)$$

tree level: $$C_0^{PS}(t) = \int d^3\vec{x}\langle 0|T\left\{\phi_{PS}(0)\phi_{PS}^\dagger(\vec{x},-t)\right\}|0\rangle$$

$$\frac{\delta C^{PS}(t)}{C_0^{PS}(t)} \xrightarrow[t>>a]{} 2\frac{\delta\left[Z_{PS}\right]}{Z_{PS}^{(0)}} - \frac{\delta M_{PS}}{M_{PS}^{(0)}} + \frac{\delta M_{PS}}{M_{PS}^{(0)}} f^{PS}(t) \qquad f^{PS}(t) \equiv M_{PS}^{(0)}\left(\frac{T}{2}-t\right)\frac{e^{-M_{PS}^{(0)}t} - e^{-M_{PS}^{(0)}(T-t)}}{e^{-M_{PS}^{(0)}t} + e^{-M_{PS}^{(0)}(T-t)}} \approx -M_{PS}^{(0)}t$$

δM_{PS} from the slope and $\delta\left[Z_{PS}\right]$ from the intercept

two further e.m. corrections due to Wilson (twisted-mass) fermions

- tadpole vertex: $$\sum_{f,\mu} e_f^2 T_\mu^f(x) = \sum_{f,\mu} e_f^2 \left[\overline{q}_f(x) \frac{i\gamma_5\tau_3 - \gamma_\mu}{2} U_\mu(x) q_f(x+\mu) + \overline{q}_f(x+\mu) \frac{i\gamma_5\tau_3 + \gamma_\mu}{2} U_\mu^\dagger(x) q_f(x) \right]$$

- shift of the critical mass: $\delta m_f^{cr} \overline{q}_f(x) i\gamma_5\tau_3 q_f(x)$

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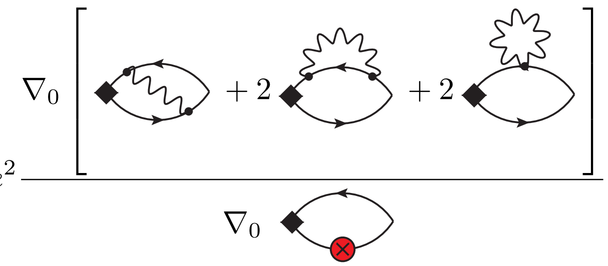
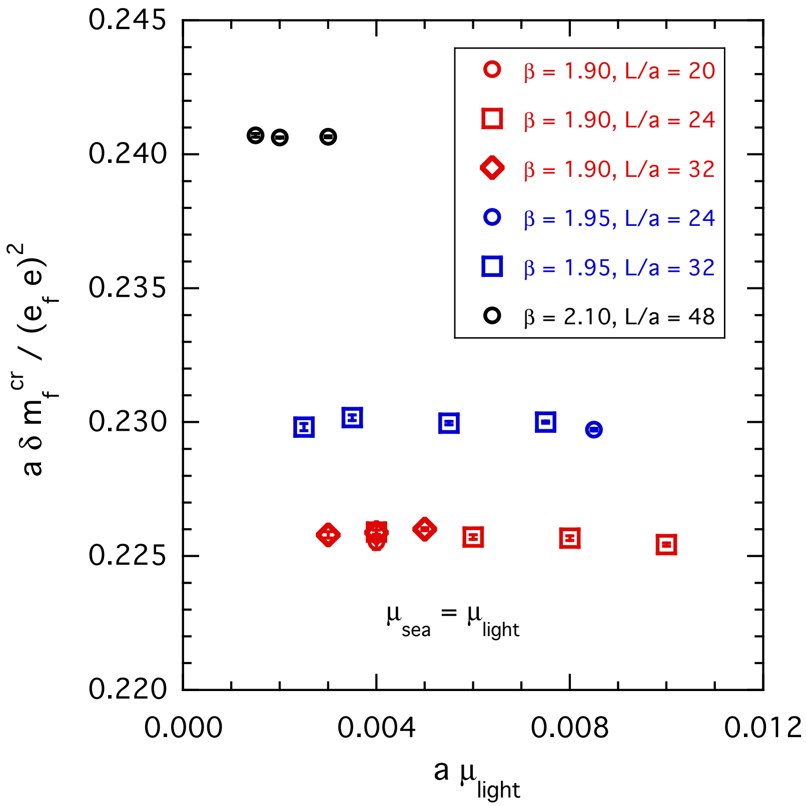
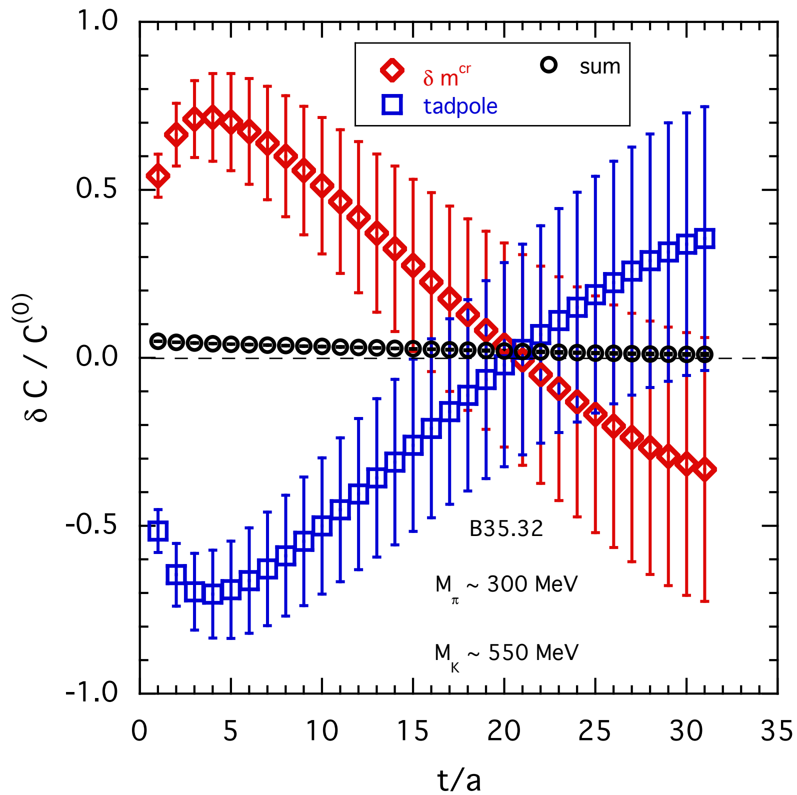
$\delta m_f^{cr} = -\frac{e_f^2}{2} e^2$



from vector WI

♦ = γ_0 ⊗ = γ_5





strong correlation between δm^{cr} and tadpole terms

the sum is well determined

besides e.m. corrections at leading order in α_{em} , we adopt the RM123 approach to evaluate **the slope of the leading strong SU(2) corrections due to $m_d \neq m_u$**, based on the insertion of the (isovector) scalar density in the isospin symmetric QCD limit

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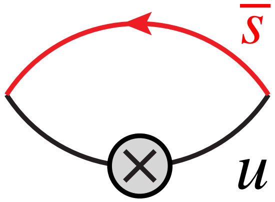
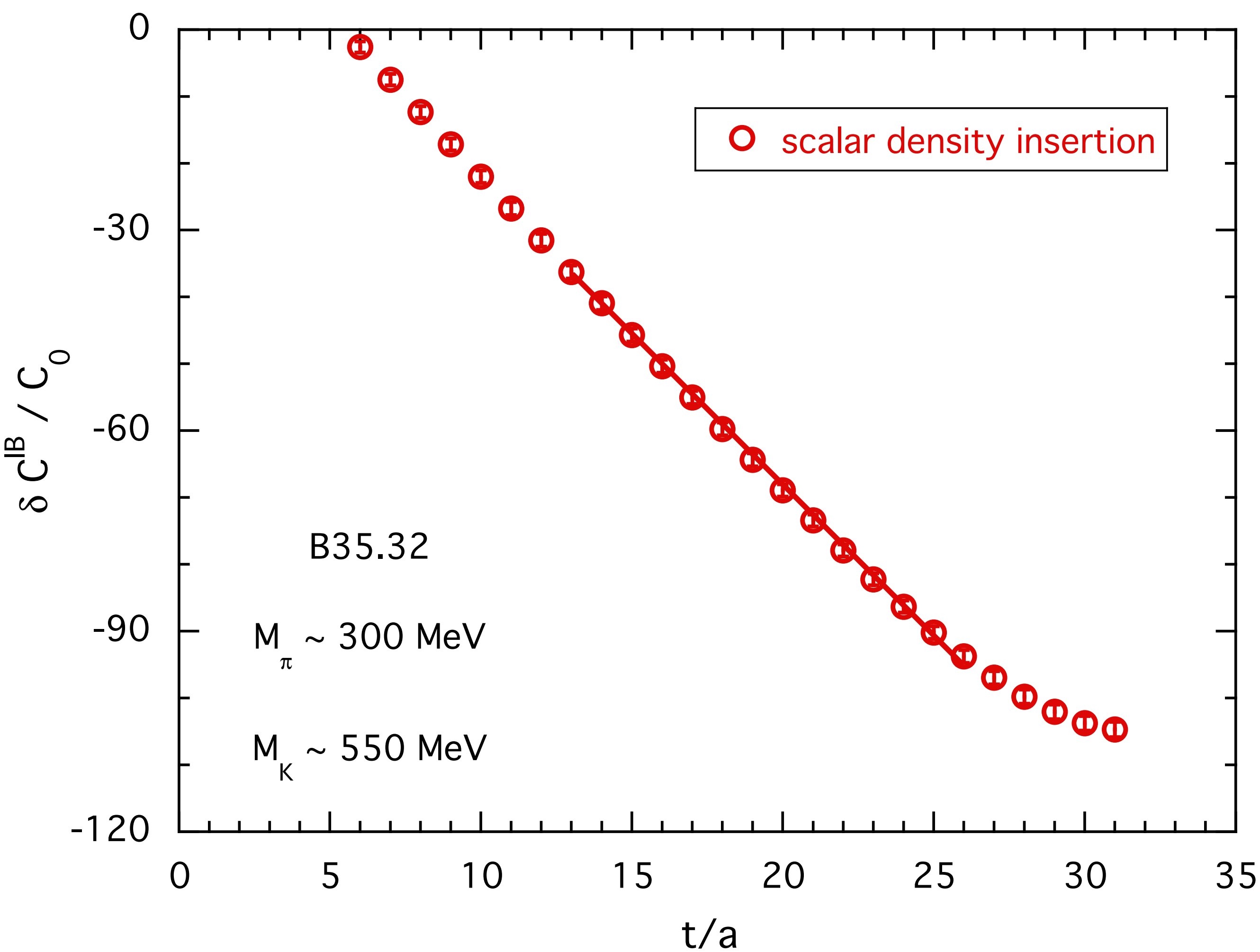
K+: 

$$\otimes = \bar{d}(x)d(x) - \bar{u}(x)u(x)$$

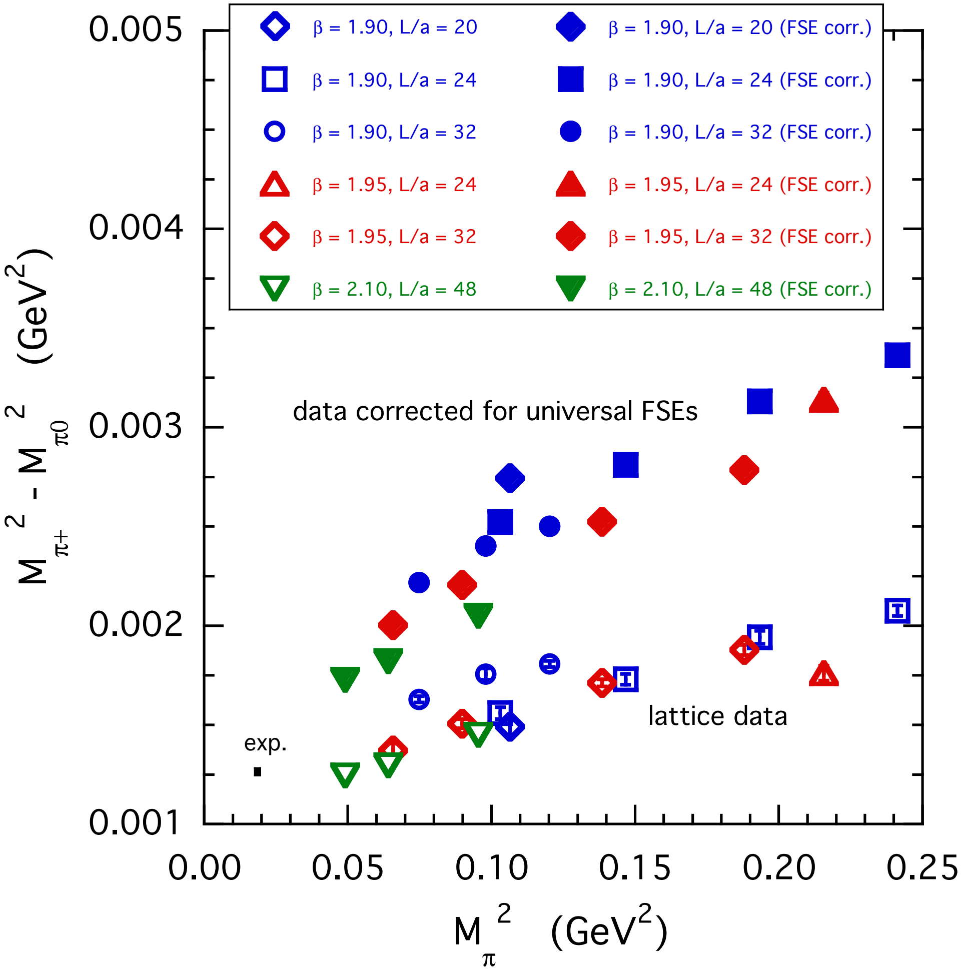


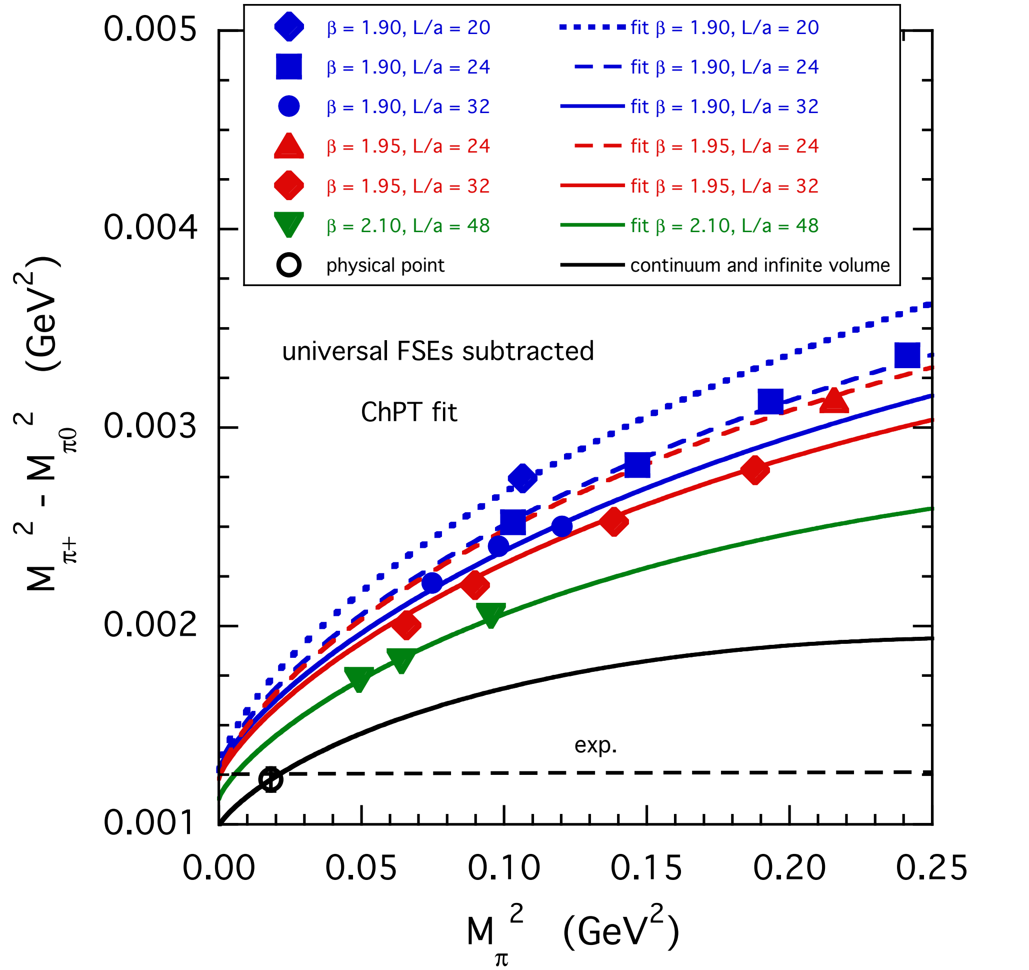
$$M_{PS} = M_{PS}^{(0)} + (m_d - m_u)\,\delta_{SU(2)} M_{PS}$$

$$A_{PS} = A_{PS}^{(0)} + (m_d - m_u)\,\delta_{SU(2)} A_{PS}$$

$\delta_{SU(2)} M_{PS}$ from the slope

$\delta_{SU(2)} A_{PS}$ from the intercept(s)

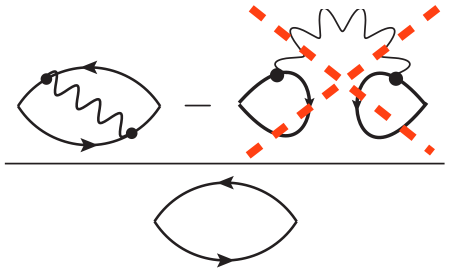




ChPT fit: Hayakawa&Uno [PTP '08]

$$\left(M_{\pi^+}^2 - M_{\pi^0}^2\right) - \alpha_{em}\frac{\kappa}{L^2}\left(2+M_\pi L\right) = \alpha_{em}4\pi f_0^2 C\left\{1-\left(4+\frac{3}{C}\right)\frac{M_\pi^2}{(4\pi f_0)^2}\left[\log\left(\frac{M_\pi^2}{\mu^2}\right)+B(\mu)\right]+Da^2+\frac{K}{L^3}\right\}$$

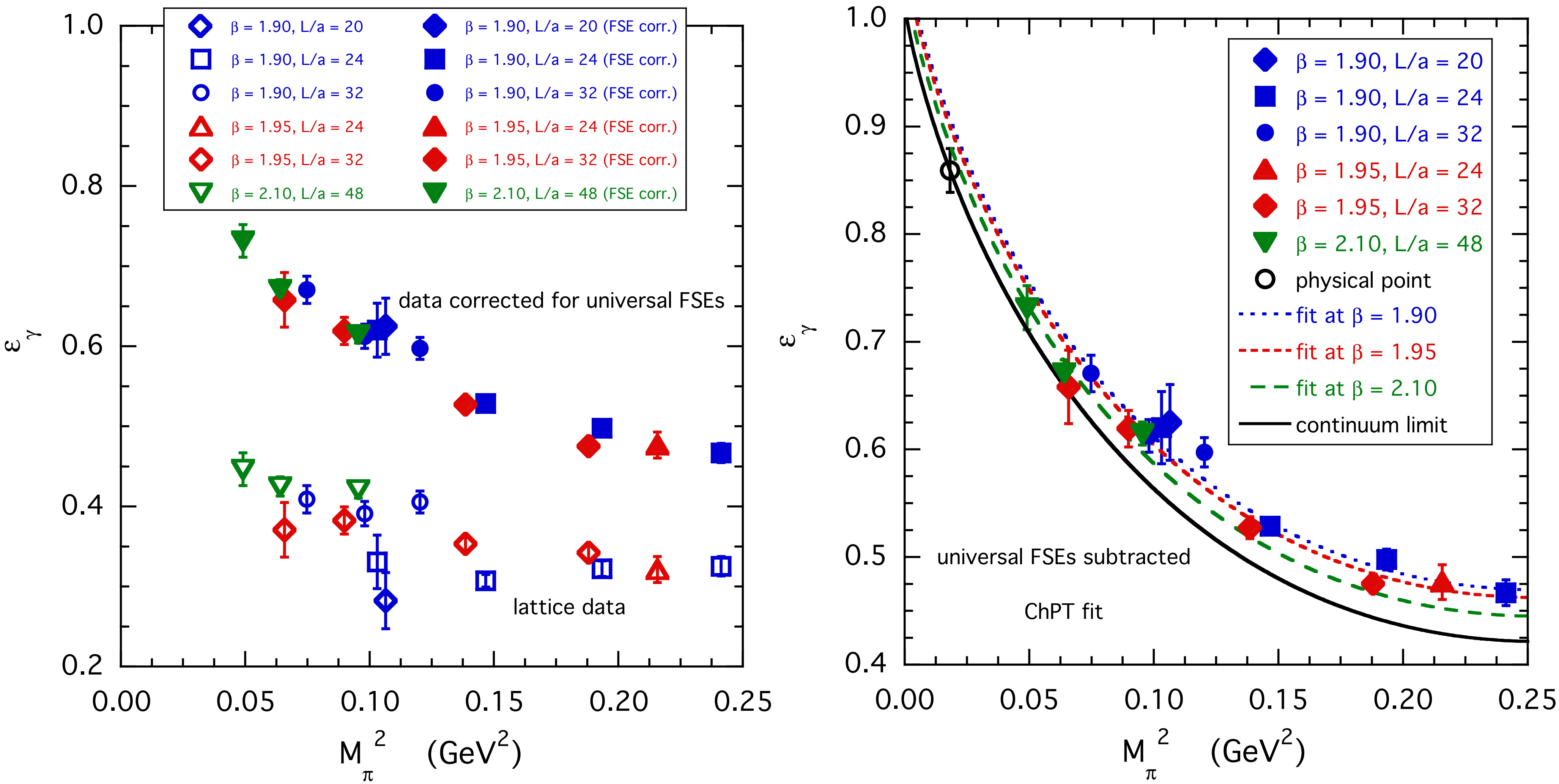
$$M_{\pi^+} - M_{\pi^0} = \frac{(e_u-e_d)^2}{2}e^2\partial_t$$



$\propto M_\pi^2$

$$\boxed{\left[M_{\pi^+}^2 - M_{\pi^0}^2\right]^{phys} = 1.226\,(58)_{stat}\,(96)_{syst}\,10^{-3}\ \mathrm{GeV}^2}$$

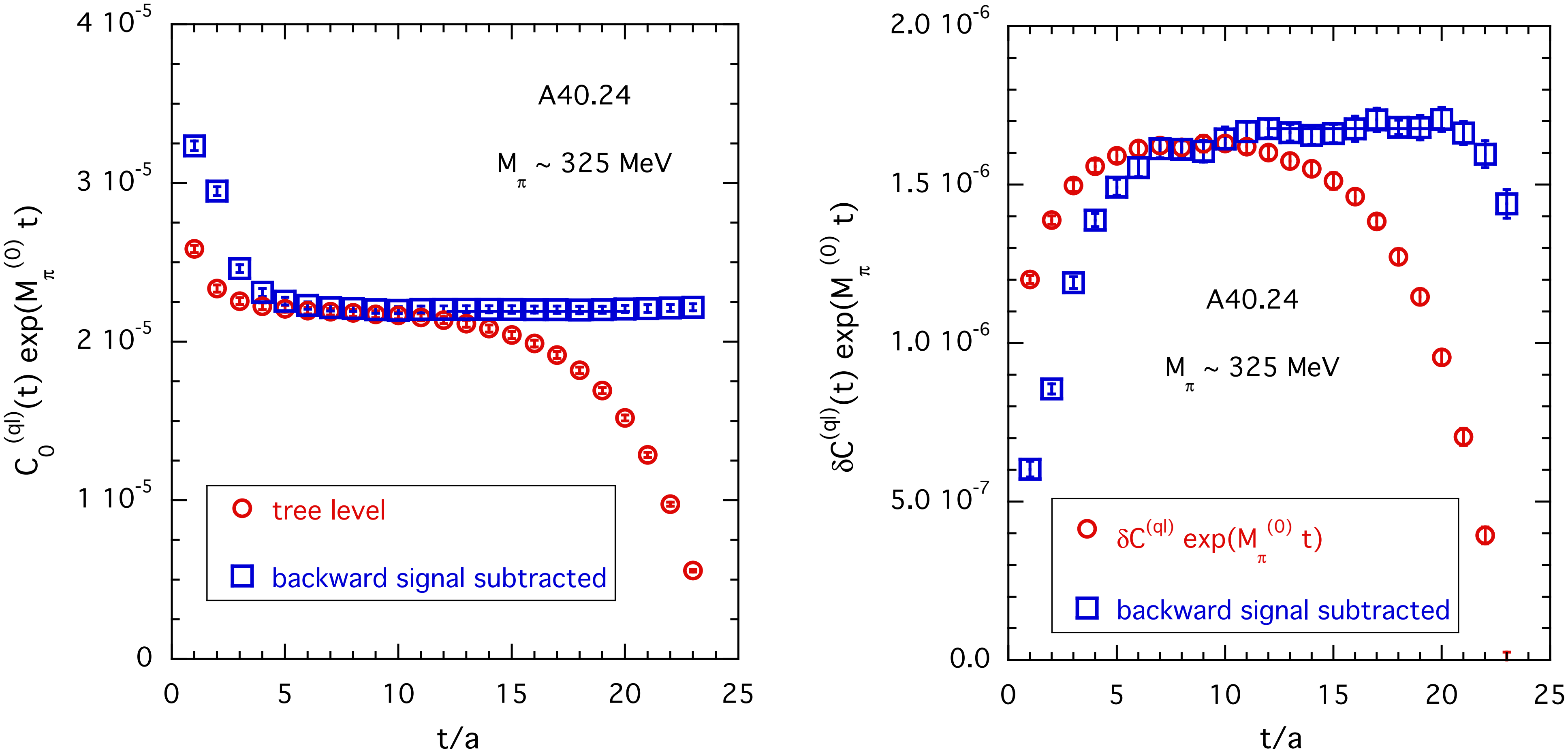
$$\left[M_{\pi^+}^2 - M_{\pi^0}^2\right]^{exp} = 1.2612\,(1)\,10^{-3}\ \mathrm{GeV}^2$$



ChPT fit: Hayakawa&Uno [PTP '08]

$$\varepsilon_\gamma - \frac{\kappa}{L}\frac{M_K - M_\pi}{4\pi f_0^2 C} = \left(\frac{4}{3} + 2Q^{sea} + \frac{3}{C}\right)\left\{\tilde{A} + \frac{M_\pi^2}{(4\pi f_0)^2}\left[\log\left(\frac{M_\pi^2}{\mu^2}\right) + \tilde{B}(\mu)\right]\right\} + \tilde{D}a^2 + \frac{\tilde{K}}{L^3} \qquad \left[Q^{sea} = 0\right]$$

* subtraction of backward signals: $\bar{C}(t)e^{M_{PS}^{(0)}t} \equiv \frac{1}{2}\left[C(t)+\frac{C(t-1)-C(t+1)}{e^{M_{PS}^{(0)}}-e^{-M_{PS}^{(0)}}}\right]e^{M_{PS}^{(0)}t} \xrightarrow[t>>a]{} const.$



2-point plateau region

* after subtraction of backward signals: $\frac{\delta\bar{C}^{(q\ell)}(t)}{\bar{C}_0^{(q\ell)}(t)} \xrightarrow[t>>a]{} \frac{\delta A_{PS}^{(q\ell)}}{A_{PS}^{(0)}}$

chirality mixing

* EM corrections to the four-fermion effective theory generate UV divergencies that can be regularized by multiplying the photon propagator by $M_W^2/(M_W^2 - k^2)$ (W-regularization)

* on the lattice a perturbative matching has been calculated at LO in α_{em} **[PRD 91 (2015) 074506]** for lattice formulations breaking chiral symmetry

$$O_1^{W-reg} = O_1^{bare} + \alpha_{em} \sum_{i=1,5} Z_i \, O_i^{bare}$$

$$O_1^{bare} = \bar{q}_2 \gamma_\mu (1-\gamma_5) q_1 \bar{\nu} \gamma^\mu (1-\gamma_5) \ell$$

$$O_2^{bare} = \bar{q}_2 \gamma_\mu (1+\gamma_5) q_1 \bar{\nu} \gamma^\mu (1-\gamma_5) \ell, \qquad O_3^{bare} = \bar{q}_2 (1-\gamma_5) q_1 \bar{\nu} (1+\gamma_5) \ell$$

$$O_4^{bare} = \bar{q}_2 (1+\gamma_5) q_1 \bar{\nu} (1+\gamma_5) \ell, \qquad O_5^{bare} = \bar{q}_2 \sigma_{\mu\rho} (1+\gamma_5) q_1 \bar{\nu} \sigma^{\mu\rho} (1-\gamma_5) \ell$$

$$Z_1 = \frac{1}{4\pi}\left[\frac{5}{2}\log\left(a^2 M_W^2\right) - 8.863\right] Z_1^{QCD}$$

Wilson fermions:

$$Z_2 = \frac{1}{4\pi}[0.536] Z_2^{QCD}, \qquad Z_3 = \frac{1}{4\pi}[1.607] Z_3^{QCD}$$

$$Z_4 = \frac{1}{4\pi}[-3.214] Z_4^{QCD}, \qquad Z_5 = \frac{1}{4\pi}[-0.804] Z_5^{QCD}$$

Z_i^{QCD} = non-perturbative QCD corrections $O(\alpha_s)$

* Wilson twisted-mass fermions (rotation to the physical basis) $\left[\langle 0 | O_5^{bare} | PS \rangle = 0\right]$

$$\left[O_1^{bare}\right]_{phys}^{W-reg} = \left[O_1^{bare}\right]_{phys} + \alpha_{em}\left\{ Z_1 \left[O_1^{bare}\right]_{phys} - Z_2 \left[O_2^{bare}\right]_{phys} - r\, Z_3 \left[O_3^{bare}\right]_{phys} - r\, Z_4 \left[O_4^{bare}\right]_{phys} \right\}$$

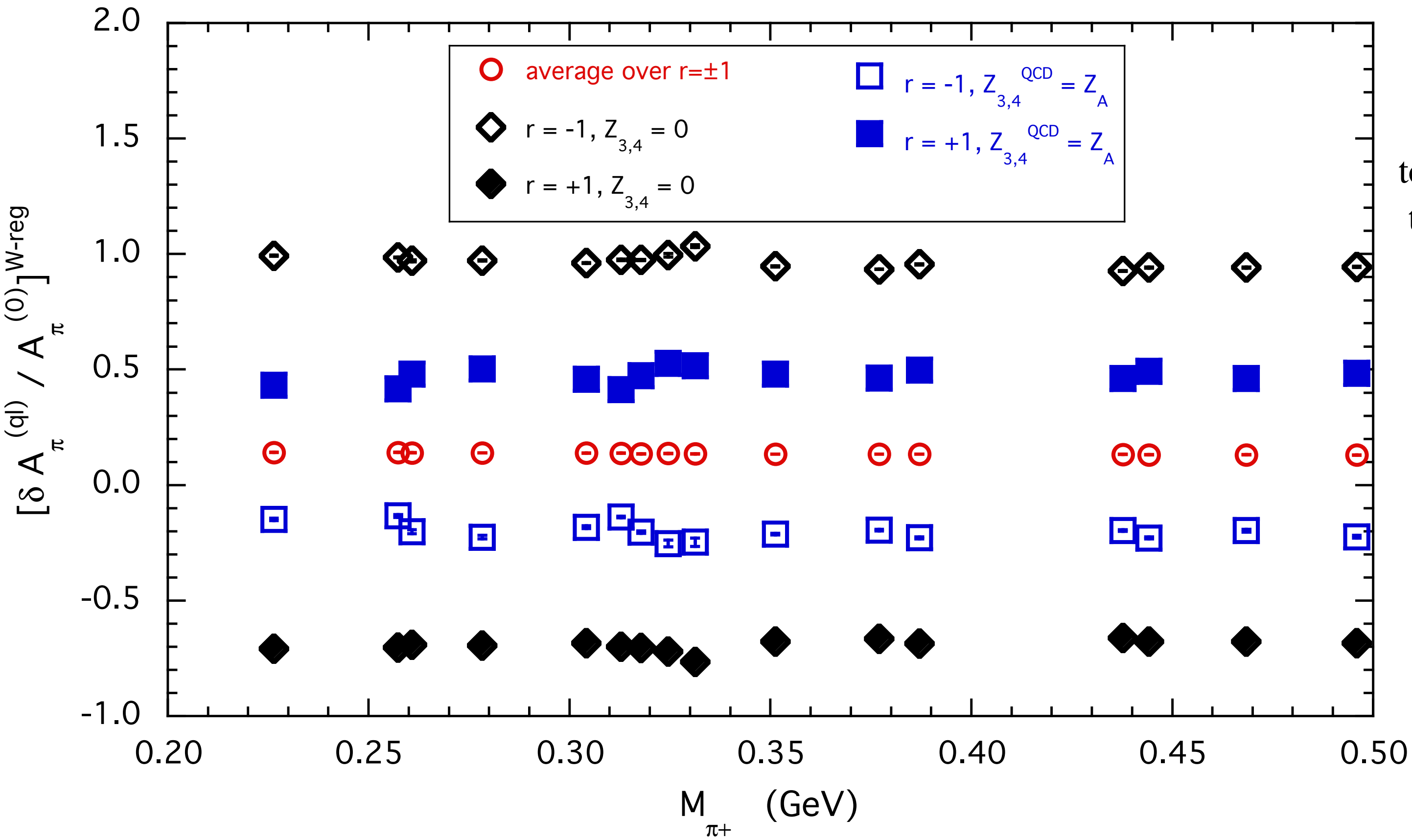
Wilson r-parameters: $r \equiv r_{q_1} \, r_\ell \qquad \left(r_{q_2} = -r_{q_1}\right) \longleftarrow$ to keep discretization errors on M_{PS} at order $O(a^2 m)$

* average over r = ±1, since physical quantities cannot depend on r

$$\delta A_{PS} = \delta A_{PS}^{qq} + \delta A_{PS}^{q\ell} + \alpha_{em}\left(Z_1 + Z_2\right) A_{PS}^{(0)}$$

$$\left[\frac{\delta A_{PS}}{A_{PS}^{(0)}}\right]^{W-reg} = \frac{\delta A_{PS}}{A_{PS}^{(0)}} + \frac{Z_1+Z_2}{Z_V} + r\left(Z_3-Z_4\right)\frac{M_{PS}^{(0)}}{m_\ell}\frac{Z_{PS}^{(0)}}{A_{PS}^{(0)}}$$

$$\langle 0|O_1^{bare}|PS\rangle = -\langle 0|O_2^{bare}|PS\rangle = A_{PS}^{(0)}$$
$$\langle 0|O_3^{bare}|PS\rangle = -\langle 0|O_4^{bare}|PS\rangle = Z_{PS}^{(0)}$$



mixings with O_1, O_2 don't depend on r
mixings with O_3, O_4 depend on r

terms due to axial current don't depend on r
terms due to vector current do depend on r

$$Z_1^{QCD} = Z_2^{QCD} = Z_V$$
$$Z_3^{QCD} = Z_4^{QCD} = Z_A$$

"factorization approximation"
between QED and QCD
vertex corrections

mixings with O_3 and O_4 can be exactly cancelled out by averaging over r = ±1

similar result can be obtained using $Z_3^{QCD} = Z_4^{QCD} \sim 0.7\, Z_A$ → 30 % violation of the "factorization approximation"

subleading effect ($\sim 10^{-3}$) in pion decay and totally absent in the decay ratio K/**π**

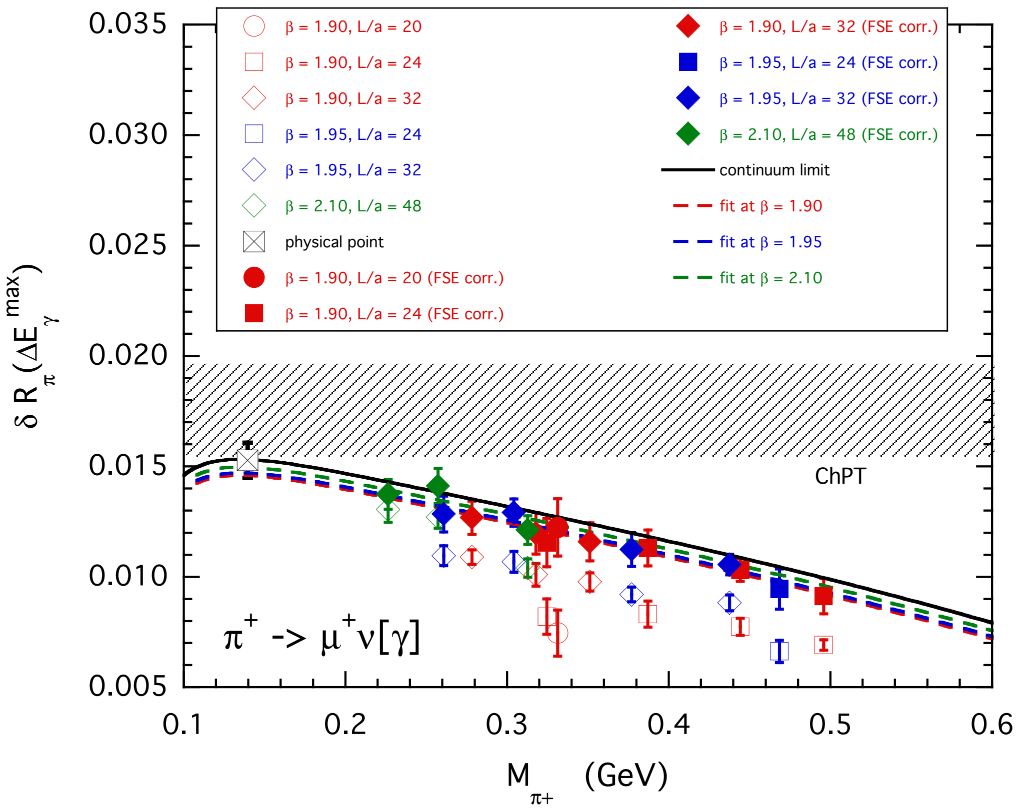
* the non-perturbative determination of Z_1^{QCD} and Z_2^{QCD} is in progress

$$\pi^+ \to \mu^+ \nu[\gamma]$$

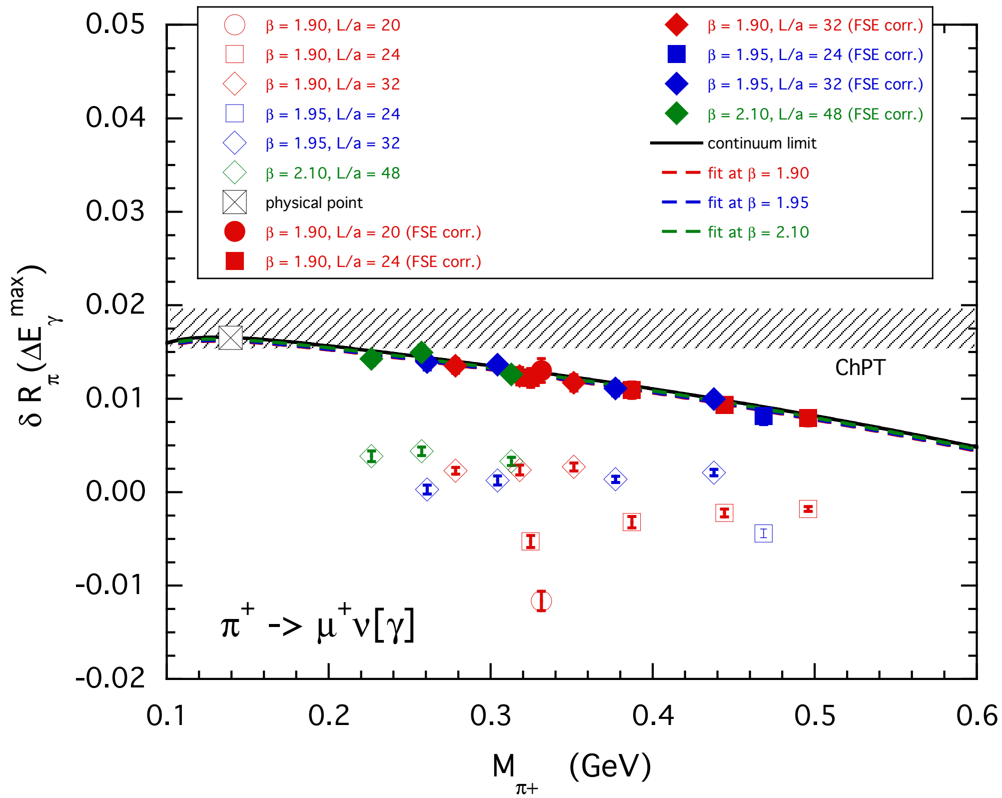
$$R_\pi\left(\Delta E_\gamma^{\max}\right) = 4\pi E(\mu) + \frac{3}{4\pi}\log\left(\frac{\xi}{\mu^2}\right) + A_1\xi + Da^2 + \delta\Gamma^{pt}\left(\Delta E_\gamma^{\max}\right) + K_\pi^{FSE}(L)$$

$$K_\pi^{FSE}(L) = \frac{K_2}{(M_\pi L)^2} + \frac{K_2^\ell}{(E_\ell L)^2}$$

subtraction of universal FSEs up to $1/L$



subtraction of point-like FSEs up to $1/L^2$



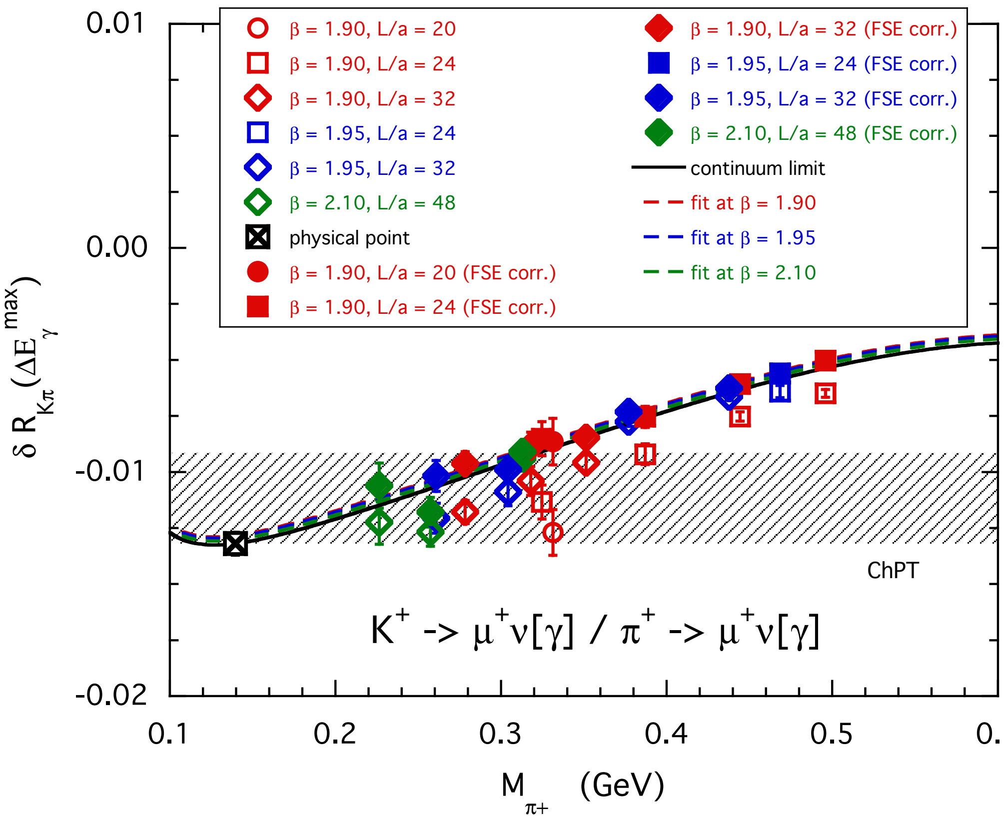
***** FSE subtraction under good control *****

$K^+ \to \mu^+ \nu[\gamma] / \pi^+ \to \mu^+ \nu[\gamma]$

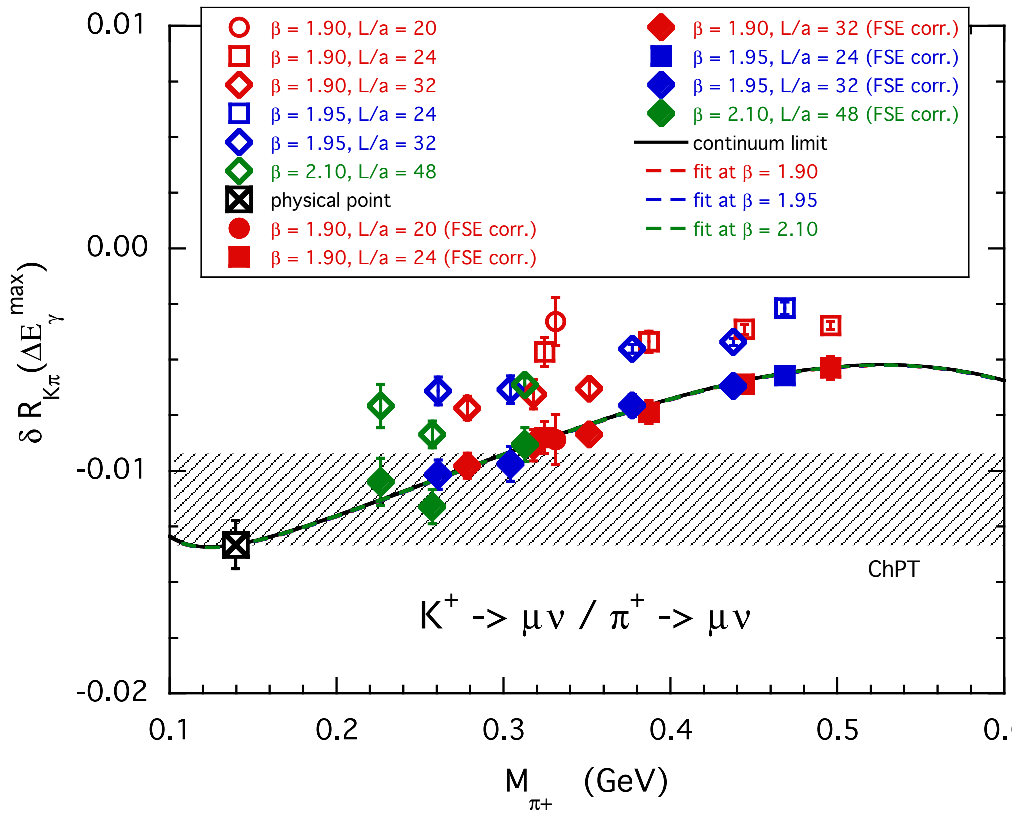
$$R_{K\pi}\left(\Delta E_\gamma^{\max}\right) = 1 + \alpha_{em}\left\{\tilde{A}_0 - \frac{3}{4\pi}\log\left(\frac{M_\pi^2}{M_K^2}\right) + \tilde{A}_1\xi + \tilde{A}_2\xi^2 + \tilde{D}a^2 + \delta\Gamma_K^{pt}\left(\Delta E_\gamma^{\max}\right) - \delta\Gamma_\pi^{pt}\left(\Delta E_\gamma^{\max}\right) + K_{K\pi}^{FSE}(L)\right\}$$

$$K_{K\pi}^{FSE}(L) = \frac{\tilde{K}_2}{(M_K L)^2} + \frac{\tilde{K}_2^\ell}{\left(E_\ell^{(K)} L\right)^2} - \frac{K_2}{(M_\pi L)^2} - \frac{K_2^\ell}{\left(E_\ell^{(\pi)} L\right)^2}$$

subtraction of universal FSEs up to $1/L$



subtraction of point-like FSEs up to $1/L^2$



***** FSE subtraction under good control *****

pion and kaon/pion analyses

$$\pi^+ \to \mu^+ \nu[\gamma]$$

data set	chiral log	a^2-term	χ^2/d.o.f.	R_π^{phys}
$b_2 = b_3 = 0$	yes	yes	0.72	0.0153 (8)
	no	yes	0.75	0.0175 (8)
	yes	no	0.74	0.0148 (7)
	no	no	0.77	0.0171 (7)
$b_3 = 0$	yes	yes	1.00	0.0165 (8)
	no	yes	0.99	0.0188 (8)
	yes	no	0.95	0.0163 (7)
	no	no	0.94	0.0185 (7)

$$\begin{aligned} R_\pi^{phys}\left(\Delta E_\gamma^{\max}\right) &= 0.0169\,(8)_{stat+fit}\,(11)_{chiral}\,(7)_{FSE}\,(2)_{a^2} \\ &= 0.0169\,(8)_{stat+fit}\,(13)_{syst} \\ &= 0.0169\,(15) \end{aligned}$$

$$\frac{K^+ \to \mu^+ \nu[\gamma]}{\pi^+ \to \mu^+ \nu[\gamma]}$$

data set	chiral log	a^2-term	χ^2/d.o.f.	$R_{K\pi}^{phys}$
$b_2 = b_3 = 0$	yes	yes	1.07	-0.0132 (5)
	no	yes	1.04	-0.0144 (8)
	yes	no	0.96	-0.0130 (5)
	no	no	0.93	-0.0142 (10)
$b_3 = 0$	yes	yes	1.18	-0.0133 (11)
	no	yes	1.14	-0.0145 (13)
	yes	no	1.14	-0.0129 (17)
	no	no	1.04	-0.0143 (11)

$$\begin{aligned} R_{K\pi}^{phys}\left(\Delta E_\gamma^{\max}\right) &= -0.0137\,(11)_{stat+fit}\,(6)_{chiral}\,(1)_{FSE}\,(1)_{a^2} \\ &= -0.0137\,(11)_{stat+fit}\,(6)_{syst} \\ &= -0.0137\,(13) \end{aligned}$$

J. Rosner, S. Stone and R. Van der Water, arXiv:1509.02220 [minireview for PDG '16]

$$R_{\pi}\left(\Delta E_{\gamma}^{\max}\right)=0.0176\,(21)$$

$$R_{K\pi}\left(\Delta E_{\gamma}^{\max}\right)=-0.0069\,(17)$$ EM contribution only

... It includes the universal short-distance electroweak correction obtained by Sirlin [18], the universal long-distance correction for a point-like meson from Kinoshita [19], and corrections that depend on the hadronic structure [20]. We evaluate [it] using the latest experimentally-measured meson and lepton masses and coupling constants from the Particle Data Group [3], and taking the low-energy constants (LECs) that parameterize the hadronic contributions from Refs. [17], [21], [22]. The finite non-logarithmic parts of the LECs were estimated within the large-N_C approximation assuming that contributions from the lowest-lying resonances dominate ...

... The uncertainty is dominated by that from theoretical estimate of the hadronic structure-dependent radiative corrections, which include next-to-leading order contributions of $O(e^2 p^2_{\pi,K})$ in chiral perturbation theory [17] ...

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