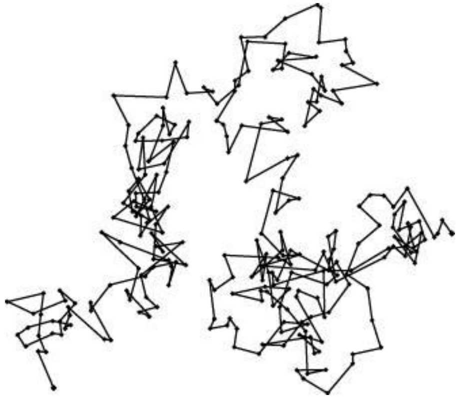
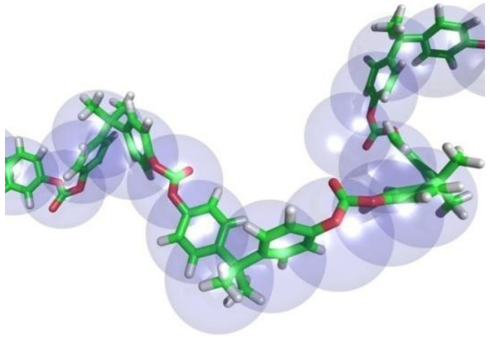
Chance Encounters: Random Walks in Science

Mustansir Barma

TIFR Centre for Interdisciplinary Sciences
Tata Institute of Fundamental Research, Hyderabad

The role of chance in diverse situations

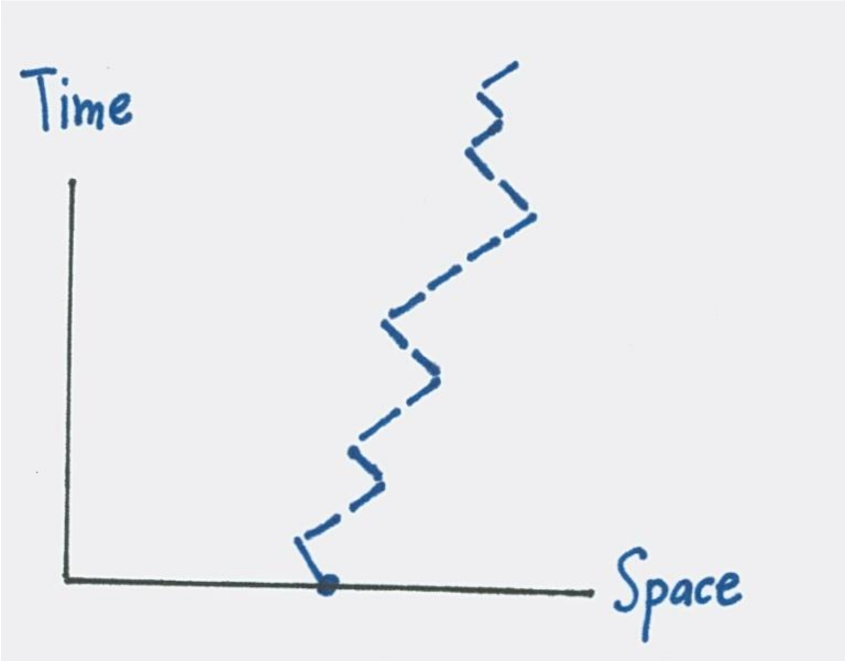




The Random Walk

A man takes one step every second,
but the step could be to the left or right.

Typical trajectory →



Questions:

How far is he likely to be after N steps?

Will he return to the starting point? If so, when?

What if the medium imposes constraints on motion?

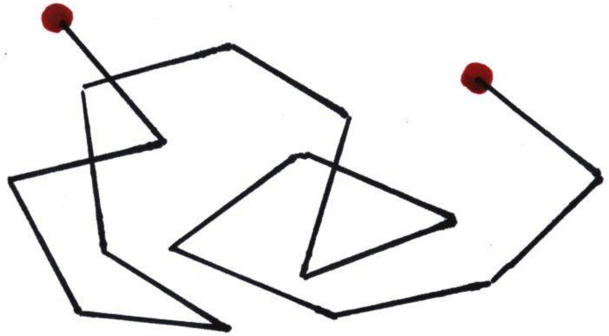
How do things change if there are many walkers?

In mathematics, an interesting problem

In physics, underlies many phenomena

In everyday life, several applications

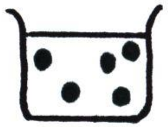
THE RANDOM WALK



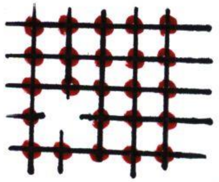
A walker takes steps in random directions

— Simplest model of random motion.

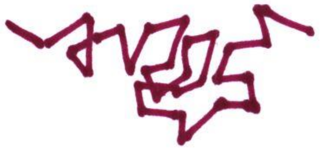
PHYSICAL REALIZATIONS



- Brownian motion



- Vacancy diffusion
- Polymer conformations



QUESTIONS

Characteristics of paths in space and time

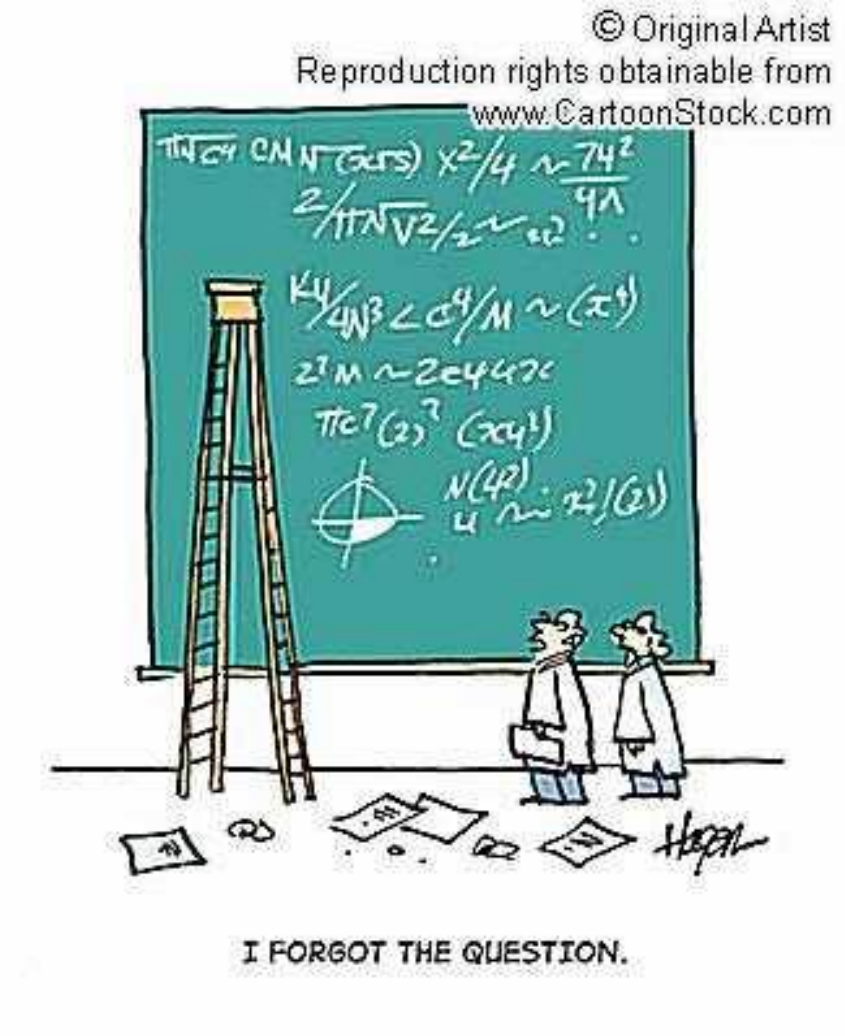
Asking the Right Question

Aims of Physics

Search for underlying laws and principles

Important : Ask the right question

'Understanding' is achieved
when we find *simple laws*
that govern *complex phenomena*



The Problem of the Random Walk

Nature, July 27, 1905

"Can any of your readers refer me to a work wherein I should find a solution of the following problem?

A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r + \delta r$ from his starting point, O."

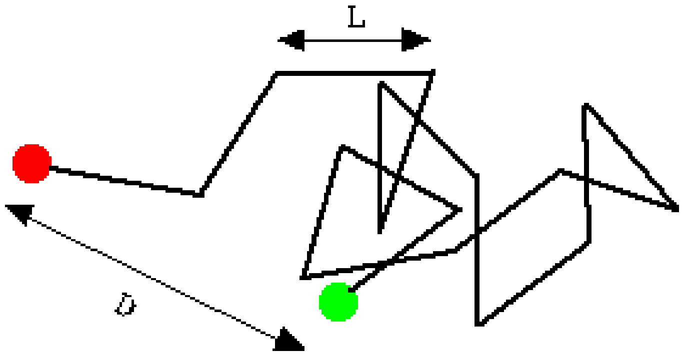
Karl Pearson

The Problem of the Random Walk

Nature, July 27, 1905

"Can any of your readers refer me to a work wherein I should find a solution of the following problem?

A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r + \delta r$ from his starting point, O."



Karl Pearson

The Problem of the Random Walk

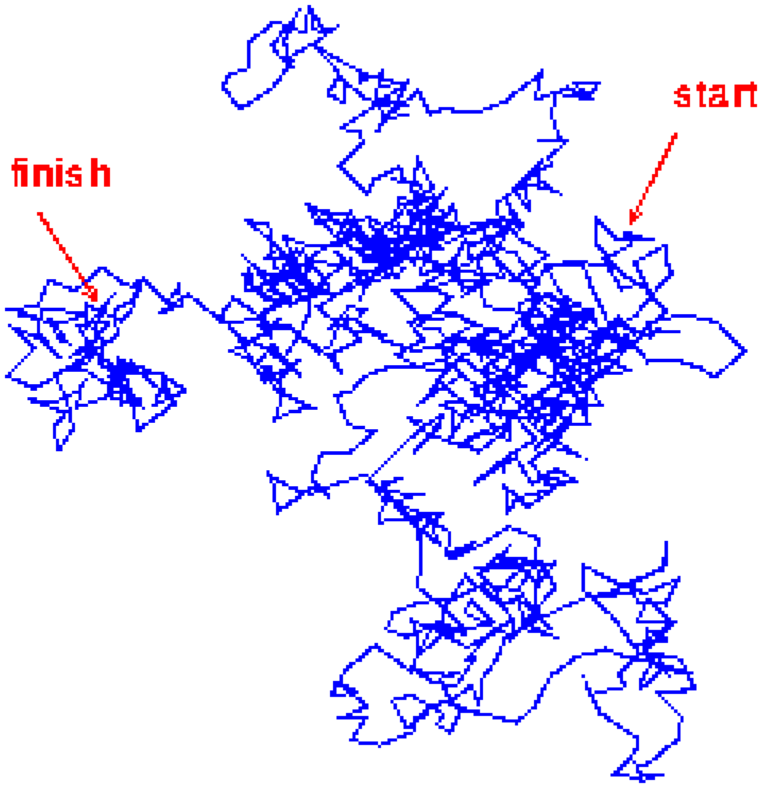
Nature, August 10, 1905

"I have to thank several correspondents for assistance in this matter … I ought to have known it, but my reading of late years has drifted into other channels, and one does not expect to find the first stage in a biometric problem provided in a memoir on sound…

The lesson of Lord Rayleigh's solution is that in open country the most probable place to find a drunken man who is at all capable of keeping on his feet is somewhere near his starting point!"

Karl Pearson

A 1000-step Random Walk



http://physics.oregonstate.edu/~landaur/nacphy/ComPhys/MONTE/FIGS/run.gif

Brownian Motion

JAN INGENHOUSE (1785)

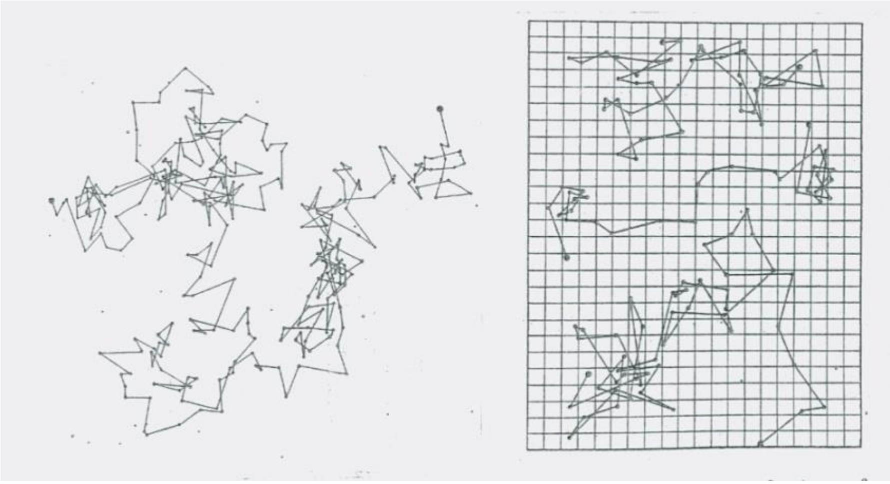
- Finely powdered charcoal on alcohol

ROBERT BROWN (1829)

- Pollen grains suspended in fluids

JEAN PERRIN (1909)

- Micron-sized colloidal particles



J. Perrin: *One may be tempted to define an average velocity of agitation by following a particle accurately … But such evaluations are grossly wrong. If this particle's positions were marked down 100 times more frequently, each segment would be replaced by a polygon as complicated as the whole drawing, and so on.*

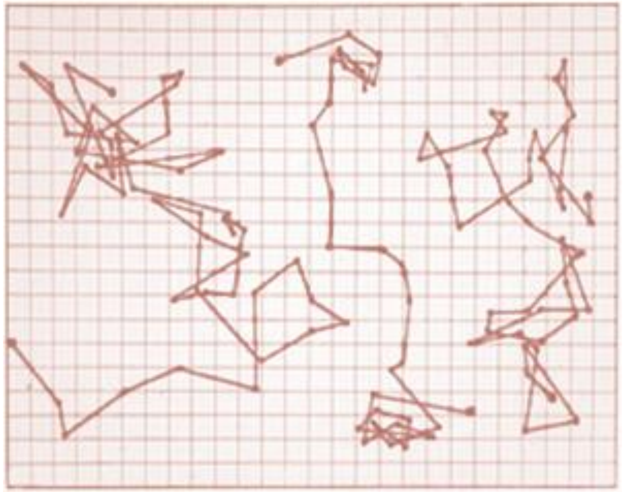
Perrin's work on Brownian motion

- established 'Molecular Reality'
- helped determine Avogadro's Number
- influenced N. Wiener → Measure

Einstein: Brownian Motion

'…a dissolved molecule differs from a suspended body *only* in size…suspended bodies should produce the same osmotic pressure as an equal number of dissolved molecules.'

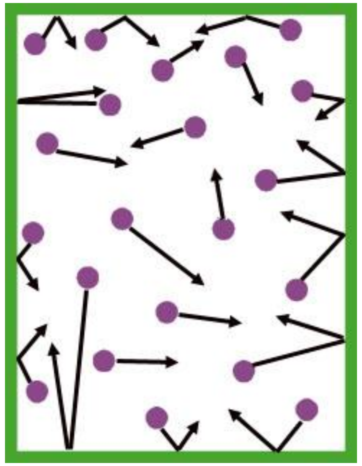
Albert Einstein's 1905 paper related Brownian diffusion to Avogadro's number



Perrin's observations of Brownian tracks

Example of Random Motion

Gas molecules in a closed container



Studied first by Kinetic Theory

(Maxwell, Boltzmann ...

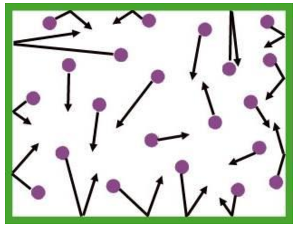
but there were others as well)

There is an interesting history.

(Several books by Stephen G. Brush)

Fluctuations of Molecular Velocities

In the 19th century, the idea that a gas contains moving molecules was not readily accepted.



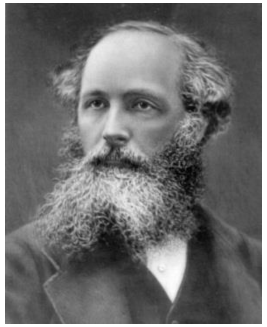
Early practitioners did not appreciate that there are *fluctuations in speeds*.

- **Clausius (1857, 1858)** calculated the pressure arising from collisions with the wall, and introduced the idea of mean free path.

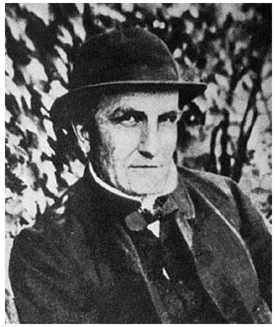
- **Maxwell (1859)** introduced the statistical approach in kinetic theory.

- **But several years earlier, Waterston (1845)** had identified the temperature with the mean squared velocity, and deduced the equipartition of energy among molecules with different masses.

Rudolf Clausius (1822 – 1888)



James Clerk Maxwell (1831–1879)



John James Waterston (1811 – 1883)

The Bombay Connection with Kinetic Theory

- Waterston was employed by the East India Company and stationed in Bombay. He pursued reading and research in the library of Grant College.

- His manuscript on kinetic theory was sent to the Royal Society in 1845. But it was rejected as 'nothing but nonsense'.

- Lord Rayleigh resurrected it 50 years later, and published it in the first issue of 'Philosophical Transactions'.

PHILOSOPHICAL TRANSACTIONS.

I. *On the Physics of Media that are Composed of Free and Perfectly Elastic Molecules in a State of Motion.*

By J. J. WATERSTON.

Communicated by Captain BEAUFORT, *R.N., F.R.S., &c.*

Received December 11, 1845,—Read March 5, 1846.

[PLATES 1, 2.]

Introduction by Lord RAYLEIGH, *Sec.R.S.*

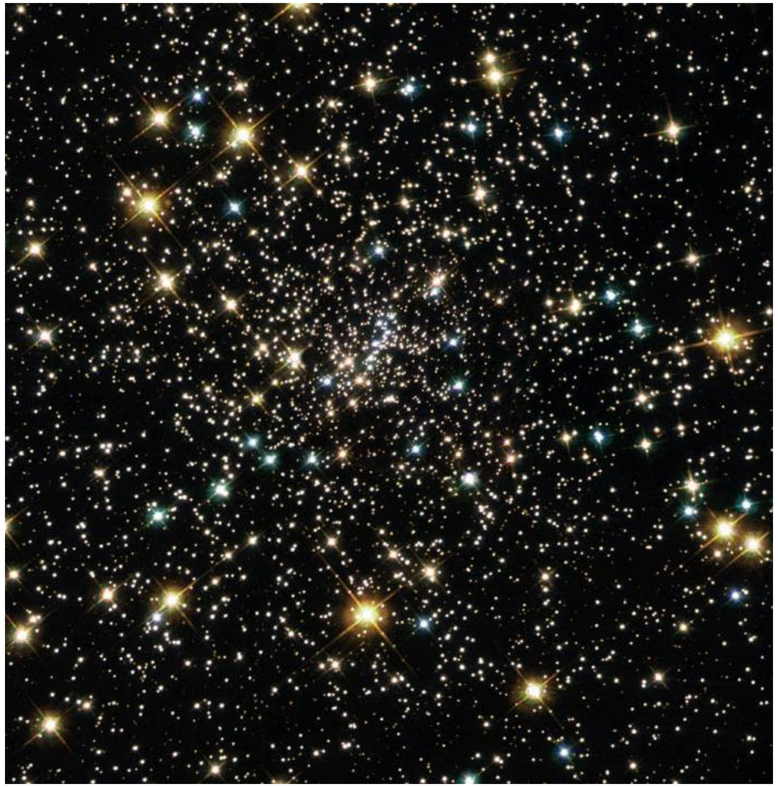
THE publication of this paper after nearly half a century demands a word of explanation; and the opportunity may be taken to point out in what respects the received theory of gases had been anticipated by WATERSTON, and to offer some suggestions as to the origin of certain errors and deficiencies in his views.

So far as I am aware, the paper, though always accessible in the Archives of the Royal Society, has remained absolutely unnoticed. Most unfortunately the abstract printed at the time ('Roy. Soc. Proc.,' 1846, vol. 5, p. 604; here reprinted as Appendix I.), gave no adequate idea of the scope of the memoir, and still less of the nature of the results arrived at. The deficiency was in some degree supplied by a short account in the 'Report of the British Association' for 1851 (here reprinted as Appendix II.), where is distinctly stated the law, which was afterwards to become so famous, of the equality of the kinetic energies of different molecules at the same temperature.

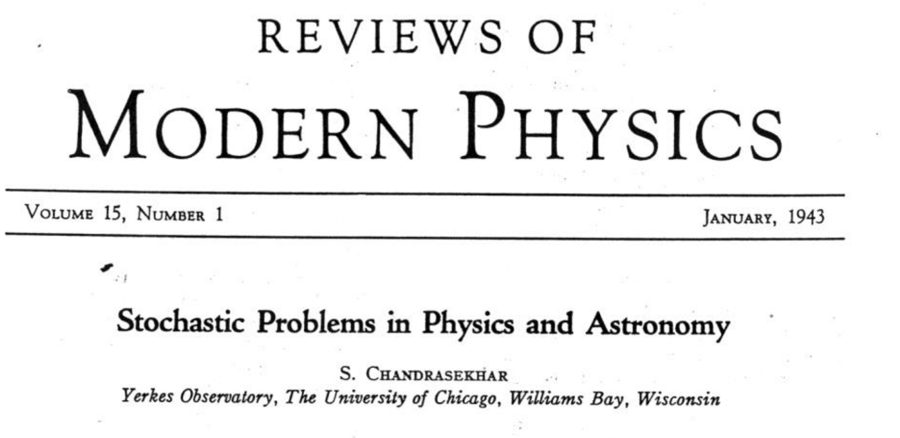
My own attention was attracted in the first instance to WATERSTON's work upon the connection between molecular forces and the latent heat of evaporation, and thence to a paper in the 'Philosophical Magazine' for 1858, "On the Theory of Sound." He there alludes to the theory of gases under consideration as having been started by HERAPATH in 1821, and he proceeds :—

Subrahmanyan Chandrasekhar

whose review article on randomness in physics and astronomy has had a great influence



http://zebu.uoregon.edu/2003/ph122/cls3.jpg



REVIEWS OF

MODERN PHYSICS

VOLUME 15, NUMBER 1 JANUARY, 1943

Stochastic Problems in Physics and Astronomy

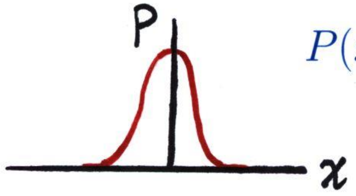
S. CHANDRASEKHAR

Yerkes Observatory, The University of Chicago, Williams Bay, Wisconsin

SOME PROPERTIES OF RANDOM WALKS

Define $D = \frac{(\text{Step length})^2}{2(\text{step time})}$

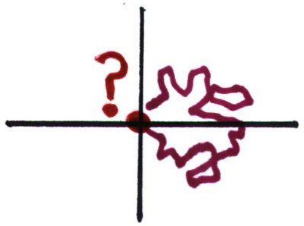
DISTRIBUTION OF DISPLACEMENT



$$P(x,t) \to \sqrt{\frac{1}{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

RMS displacement $= \sqrt{2Dt}$

PROBABILITY OF RETURN (Polya)

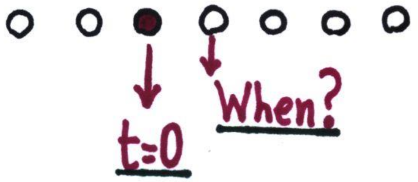


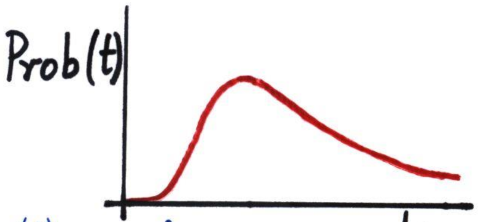
Probability $= 1$ **if** $d = 1, 2$

< 1 **if** $d \geq 3$

($\simeq 0.34$: **simple cubic lattice**)

MEAN TIME TO REACH A SITE





$\langle t \rangle = \infty!$

(Gambler's Ruin)

Gambler's Ruin: *How long to recover a loss?*

Coin-tossing game → Random walk

(TTHTHTHTHH)→(LLRLRLRLRR)

www.blueworkhorse.com/articles/nfl/manning-si...

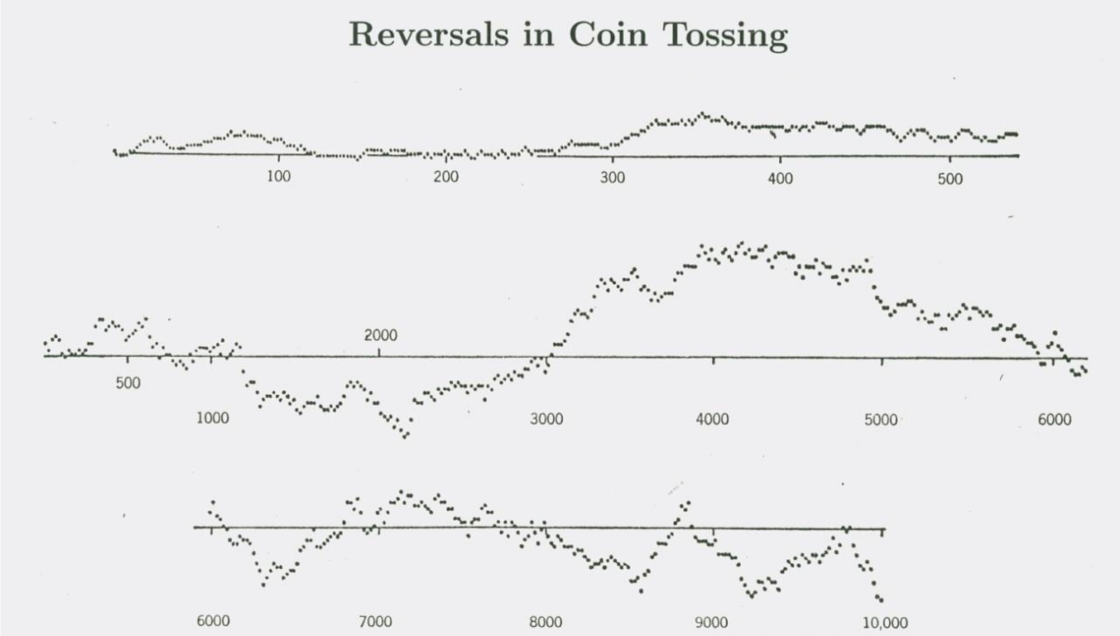
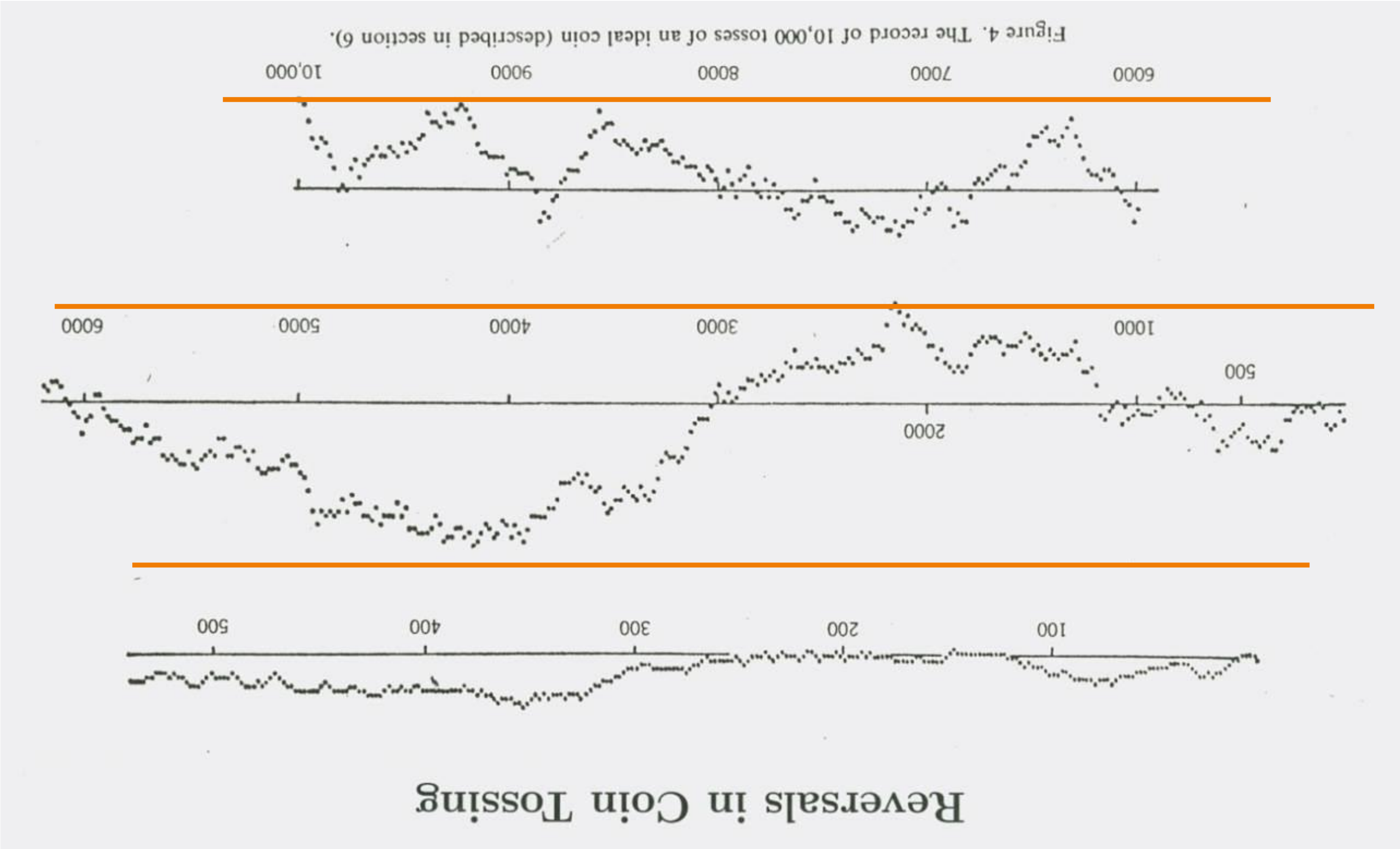


Figure 4. The record of 10,000 tosses of an ideal coin (described in section 6).

Even trained statisticians expect much more than 78 changes of sign in 10,000 trials, and nobody counted on the possibility of only 8 changes of sign … theoretically, neither should cause surprise. If they seem startling, this is due to our faulty intuition and to our having been exposed to too many vague references to a mysterious "law of averages".

(W. Feller, `Introduction to Probability Theory & its Applications')

Gambler’s Ruin: *The reverse sequence*



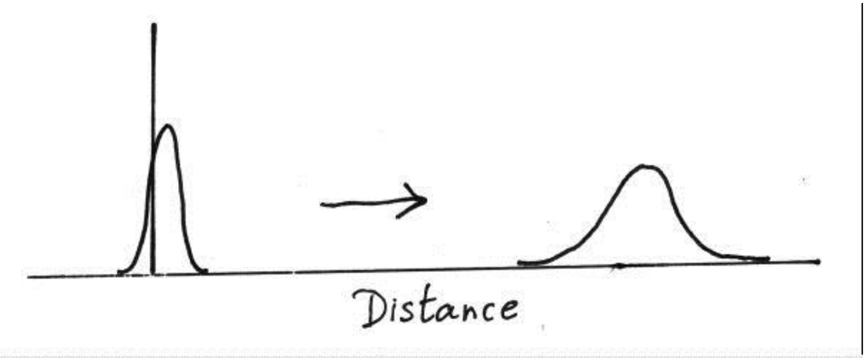
Biased Random Walks

Rightward stepping rate u (step length $= a$)

Leftward stepping rate v

Drift velocity $c = (u - v)a$

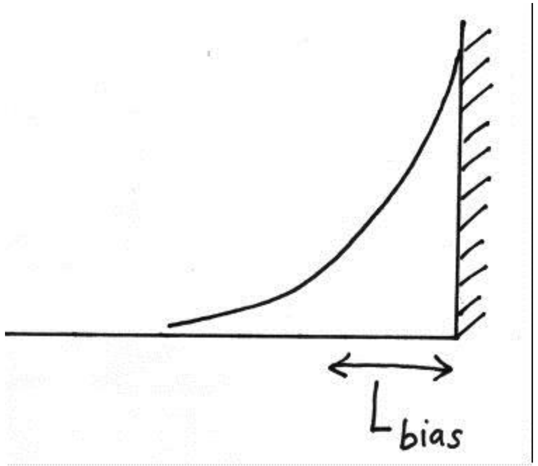
Diffusion constant $D = \sqrt{uv}\, a^2$



Diffusion length $L_{Bias} \approx D/c$

describes

- Crossover from diffusion to drift
- Density pile-up near a wall



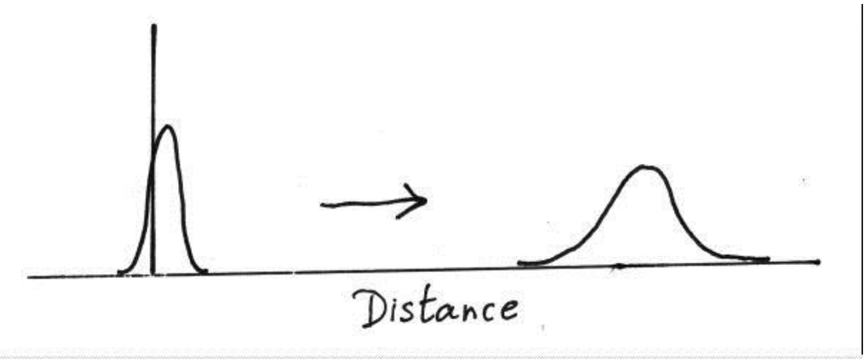
Biased Random Walks

Rightward stepping rate u

Leftward stepping rate v

Drift velocity $c = (u - v)a$

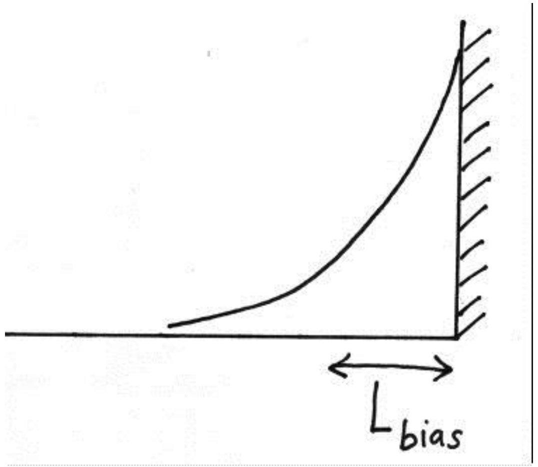
Diffusion constant $D = \sqrt{uv}\, a^2$



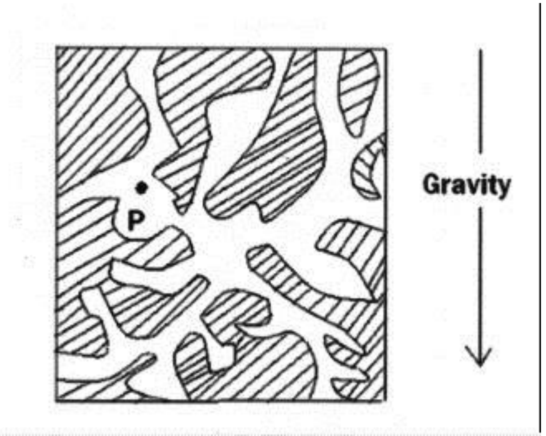
Diffusion length $L_{Bias} \approx D/c$

describes

- Crossover from diffusion to drift
- Density pile-up near a wall



What if the particle is constrained to move in a randomly disordered medium?

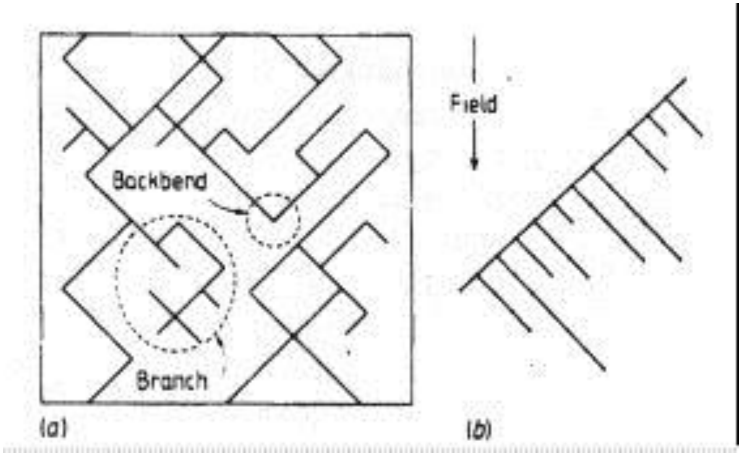


Biased Random Walks in a Disordered Medium

There is a competition between Drift and Trapping

$\downarrow$ $\downarrow$

Important at Low Bias High Bias



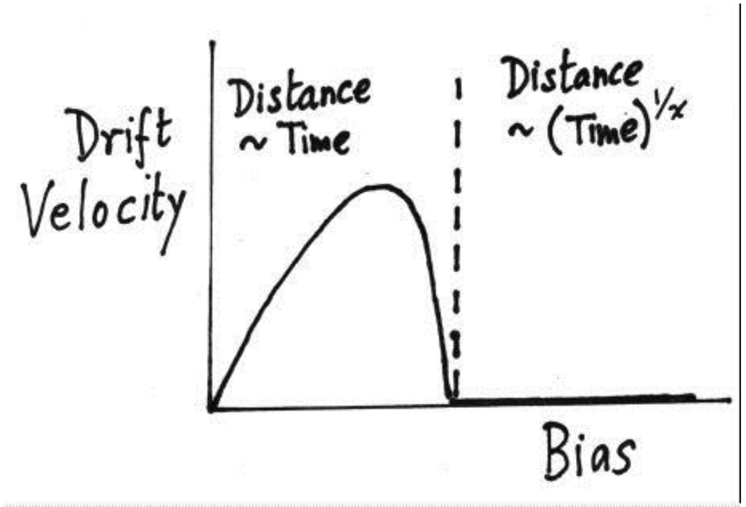
Two exponentials compete:

Prob (branch depth $= h) \sim \exp(-\frac{h}{\xi})$

Mean exit time from branch of depth $h \sim \exp(\frac{h}{L_{Bias}})$

Crucial parameter $x = L_{Bias}/\xi$

$x < 1 \rightarrow$ Anomalous transport

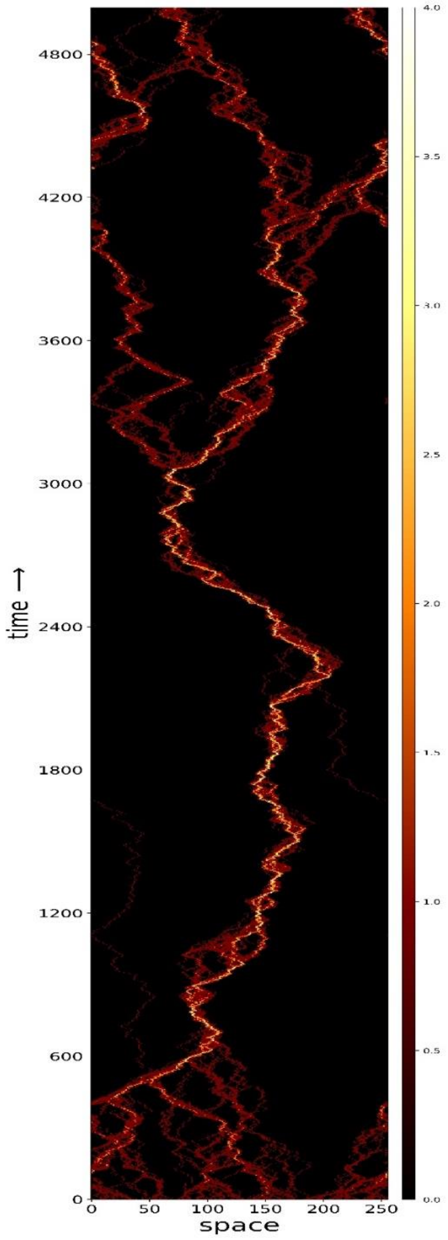


Example:

For a sedimenting particle, the bias depends on T. With a 0.3 micron metal particle in a medium with 3 micron dead-ends, $T_{critical}$ is 330 K.

M. Barma, D. Dhar (1982); S. R. White, M. Barma (1984); V. Balakrishnan, C. van den Broeck (1995); T. Demaerel, C. Maes (2018)

Walkers Driven by Fluctuating Environments



Consider diffusing particles advected by a compressible fluid or a growing, fluctuating surface

- How does the displacement $r(t)$ of a walker increase with t ?
- How does the separation $s(t)$ of two walkers increase with t ?

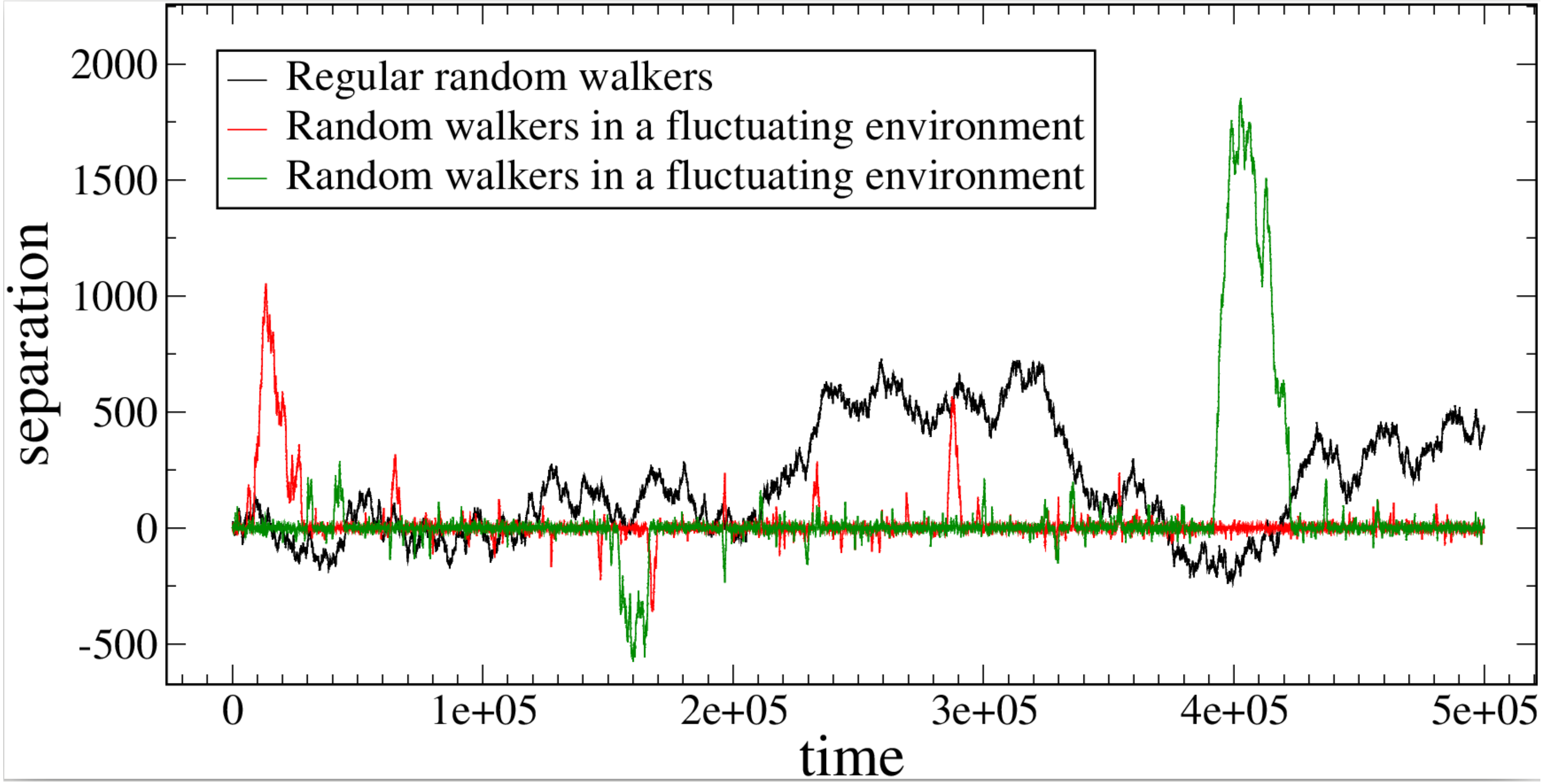
Regular Random Walkers

- $< r^2(t) > \approx 2Dt$
- $< s^2(t) > \approx 4Dt$
- $Prob(s = 0) \; \rightarrow 0 \; as \; t \rightarrow \infty$

Random Walkers in a Fluctuating Environment

- $< r^2(t) > \approx D_1 t^{4/3}$
- $< s^2(t) > \approx D_2 t$ …. *But the motion is far from diffusive!*
- $Prob(s = 0) \; \rightarrow finite \; constant \; as \; t \rightarrow \infty$

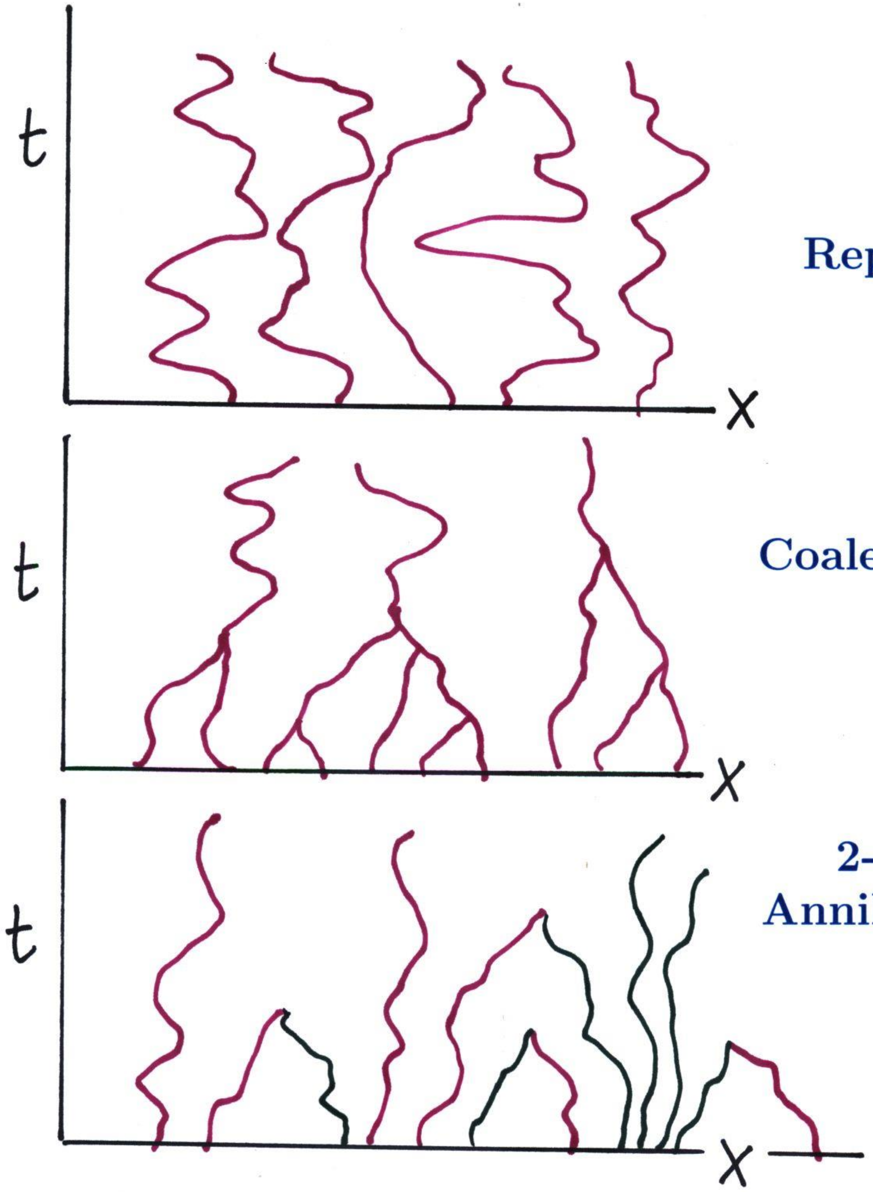
M. Ueda, S. Sasa (2015); T. Singha, M. Barma (2018)



Regular random walkers
Random walkers in a fluctuating environment
Random walkers in a fluctuating environment
2000
1500
1000
500
0
-500
separation
0
1e+05
2e+05
3e+05
4e+05
5e+05
time

INTERACTING RANDOM WALKS

Many walkers with short-ranged interactions.

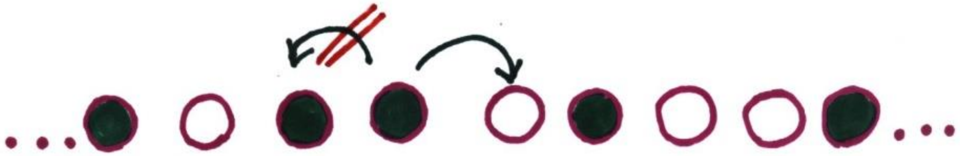


Repulsion

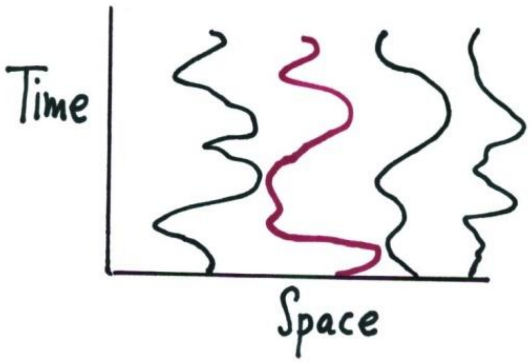
Coalescence

2-Species Annihilation

RANDOM WALKS WITH EXCLUSION



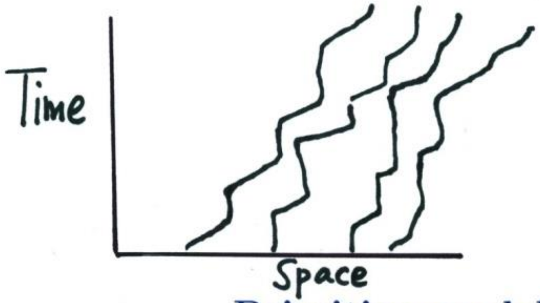
No double occupancy



$\sqrt{\langle (\text{Displacement})^2 \rangle} \sim t^{1/4}$

Caging effect

With a driving field



- **Finite Current**
- **Shock Waves**

Primitive model of traffic.

Traffic and Moving Crowds

Time-exposed photographs of traffic and crowds

www.unfinishedman.com

olsonfarlow.com

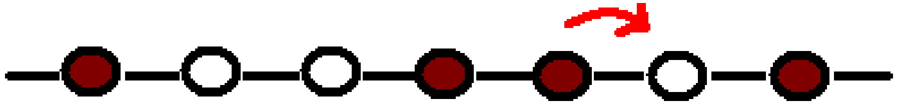
These are examples of **current-carrying systems**

Suggests: A physics approach may prove useful

Caution: The current carriers are rather individualistic!

Models of Traffic

Random-walk models
can be adapted to traffic

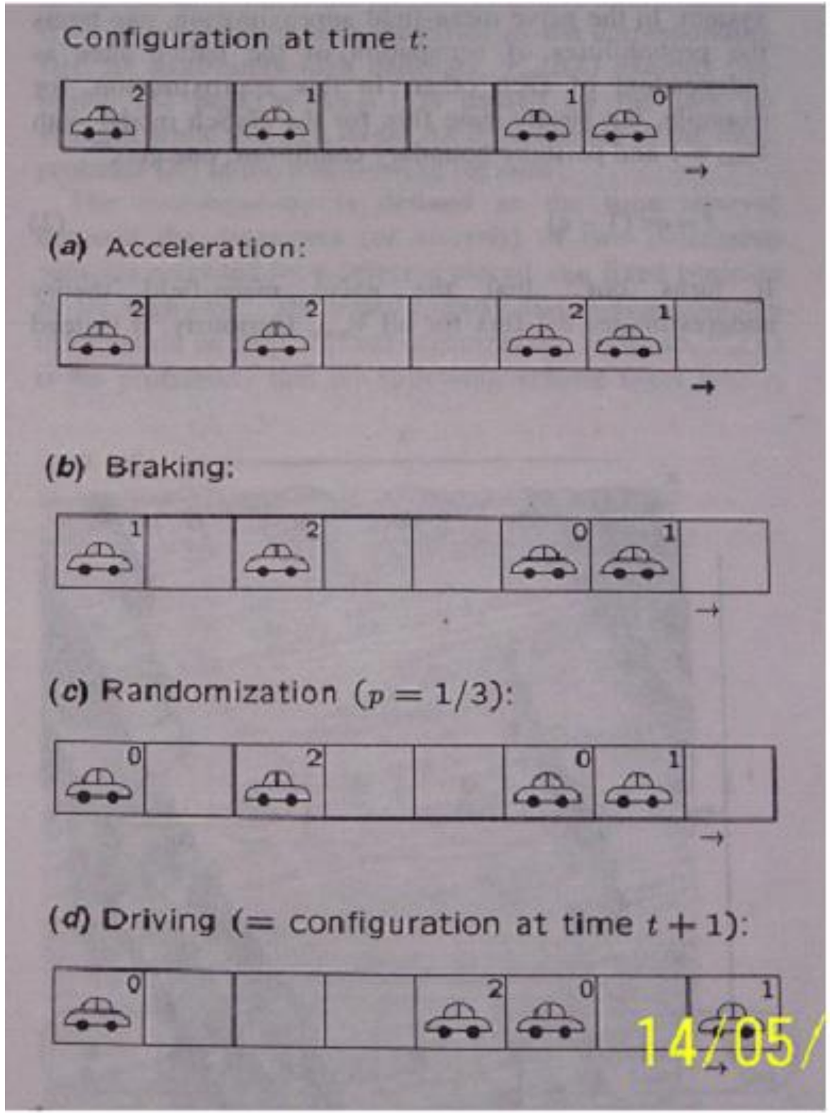


Nagel – Schreckenberg model

Road: A succession of boxes

Cars: Hopping between boxes
with 0 or 1 car per box

Crucial to incorporate unpredictability



Ref: D. Chowdhury et al, Current Science

From Models to Reality

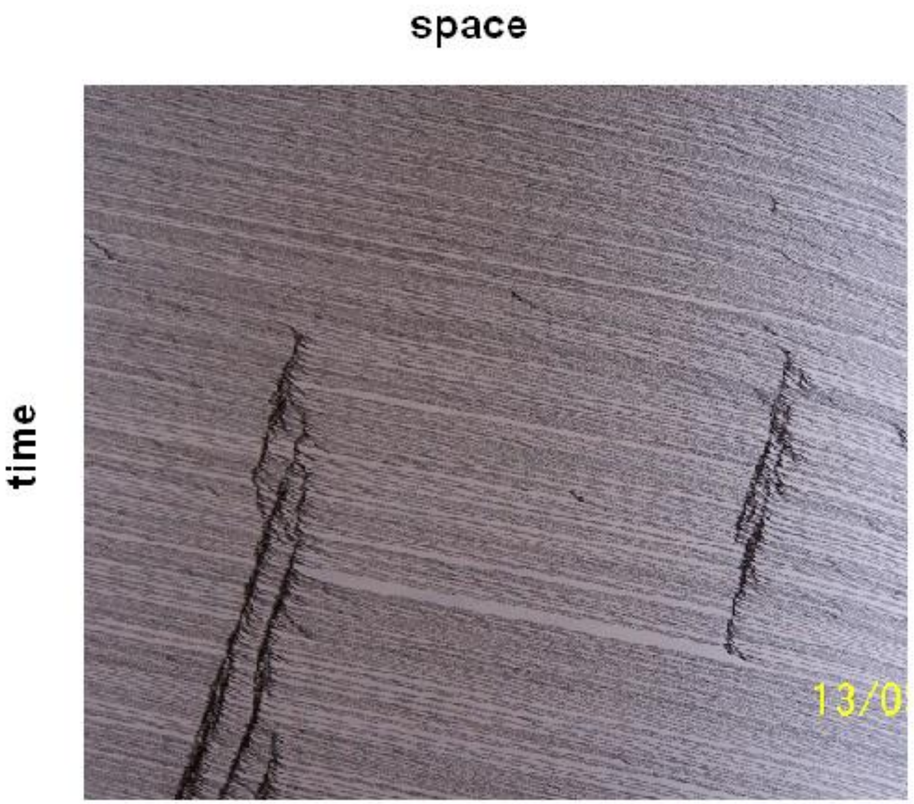
Are such models realistic ?

Yes. A similar model was used to predict traffic patterns on German highways

(Drivers could get a 1 hour forecast online)

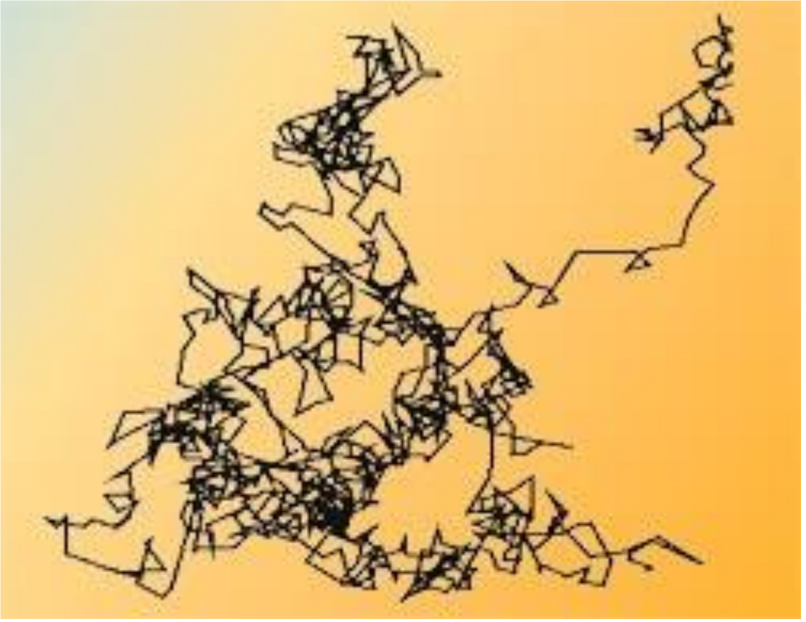
Can these models shed light on traffic jams?

Random fluctuations can lead to jams




DN 09 C 2121

Conclusions



math.tulane.edu/~xdw/cbms/cbms.html

- **Random walks**

  --- a paradigm for random motion

  --- entertaining , interesting, instructive

  --- many applications in mathematics, physics, chemistry, biology …

  --- a starting point to describe some problems in everyday life

Random Walk video clip

- Random Walk Simulation.mp4`

Colliding Particles

- 3D Visualization of Random Walk During Particle Diffusion_Cut_17_Sec.mp4