Gluon TMDs at Small x

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Review of Small-x

Small-x evolution in the target

McLerrean Venugopalan Model
(1994)

JIMWLK Equations
(1997-2001)

Color Glass Condensate
(2001)

Small-x evolution in the projectile

BFKL equation

Glauber-Gribov-Mueller
multiple-rescatterings model
(1990)

Mueller's Dipole Model
(1994-95)

Balitsky-Hierarchy (1996)

Balitsky-Kovchegov Equation
(1999)

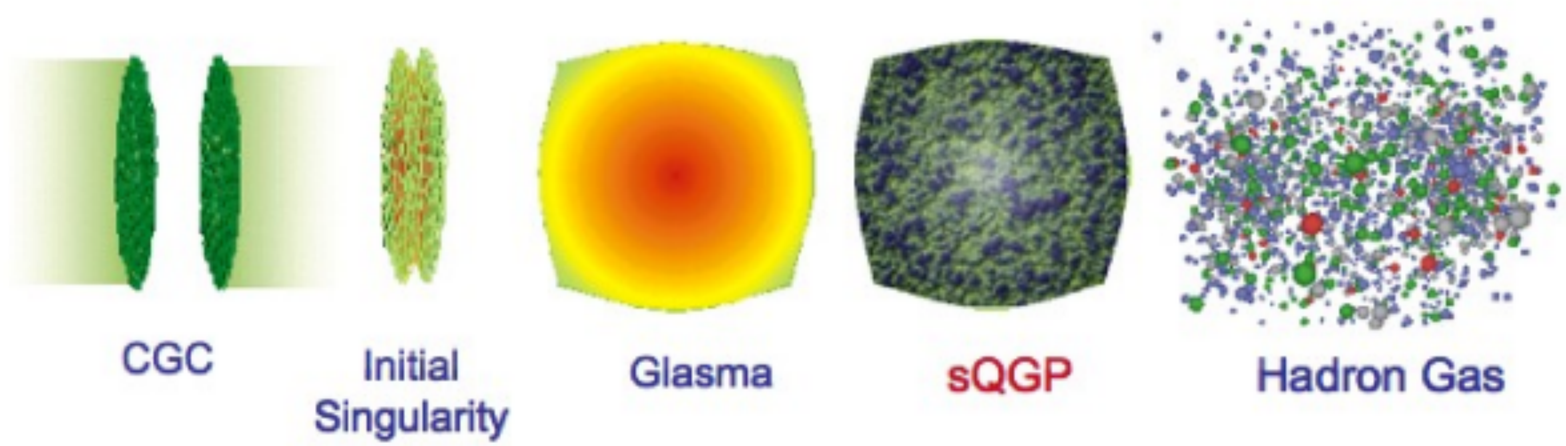
Gluon TMDs at small x

Color charge to Color field

How to build a field theory for nucleus-nucleus or proton-nucleus collision at high energy from first principle i.e. from QCD Lagrangian?

Unsettled issued / Open problems

- Why thermalisation is so rapid?
- Why hydrodynamics is so successful?
- How to get initial conditions?



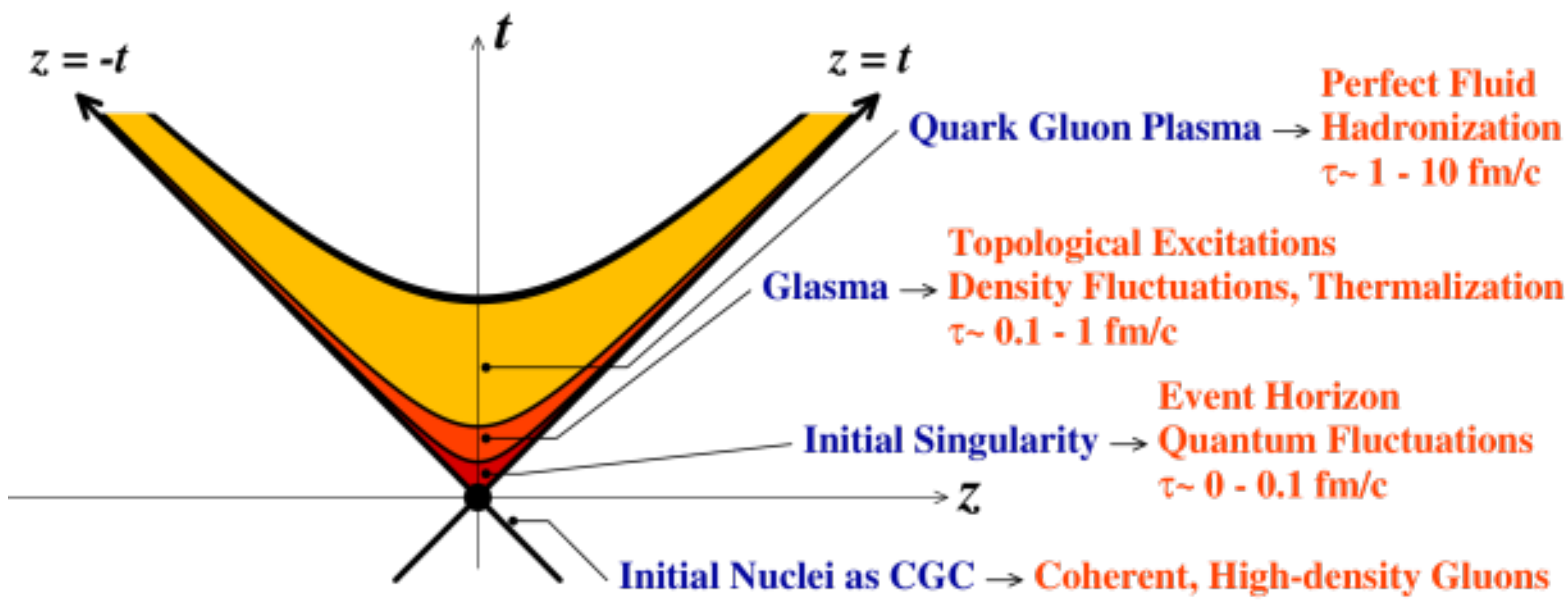
Color Charge
Color Current

Color Field

- How to model color charge density and charge current?
- How to get classical color field from color density and color current?

Color Glass Condensate and Glasma

Modelling the color density and color current



Light Cone Coordinate

- Fast moving partons inside hadron suffer time dilation.
- Fast partons do not evolve during short duration of a collision.

$$(x^0, x^1, x^2, x^3) \rightarrow (x^+, x^-, x_\perp)$$

Time

Color charge density $\rho_a\left(x^+, x^-, x_\perp\right) \rightarrow \rho_a\left(x^-, x_\perp\right)$ Time independent

Color charge current $J_a^\mu = \delta^{\mu +}\rho_a\left(x^-, x_\perp\right)$ Only longitudinal current survives

Yangs Mills equation

- How to get classical color field from color current?

Color density

Color current

$$\left[D_{\mu}^{a}, F^{\mu\nu a}\right] = J^{\mu a}$$

Color Field

$$J_{a}^{\mu} = \delta^{\mu +}\rho_{a}\left(x^{-}, x_{\perp}\right)$$

Yang-Mills equation for the color field of the target

$$D_{\mu}^{a} = \partial_{\mu} - igt^{a}A_{\mu}^{a}$$

$$F^{\mu\nu a} = \partial_{\mu}A_{\nu}^{a} - \partial_{\nu}A_{\mu}^{a} + gf^{abc}A_{\mu}^{b}A_{\nu}^{c}$$

Solution in Lorentz gauge

$$A^{-} = A^{\perp} = 0$$

$$A^{+}(x) = -\frac{1}{\nabla_{\perp}^{2}}\rho_{a}(x^{-}, x_{\perp})$$

Specifying ensemble of target color filed (A) is equivalent to specifying the ensemble of charge density (ρ).

- How to specify ensemble of color charge density?

The framework must be supplemented with a probability distribution $\mathscr{W}\left[\rho\right]$

McLerrean Venugopalan Model $\langle \rho_a(x^-, x_\perp)\rho_b(x^-, x_\perp)\rangle = \mu^2 \delta^{ab}\delta(x^- - y^-)\delta^2(x_\perp - y_\perp)$

Larry D. McLerran, Raju Venugopalan (1994)

Justification

- In a highly Lorentz contracted nucleus, there is a large density of color charges at each impact parameter.
- The charges from different nucleons are uncorrelated.
- Charges in different nucleus are not correlated.

Ensemble of color charge density

- How to evaluate the observables?

Larry D. McLerran,
Raju Venugopalan (1994)

Procedure

- Randomly pick ρ_1 and ρ_2 (one for each nucleus) from the gaussian distributions $W_1[\rho_1]$ and $W_2[\rho_2]$

$$\left[D_\mu^a, F^{\mu\nu a}\right] = \delta^{\nu -}\rho_1 + \delta^{\nu +}\rho_2$$

- Solve (numerically) the Yang Mills equation with two sources
- Evaluate the observable of interest on the classical field A^μ

$$\frac{dN_g}{dx d^2k_\perp} = \frac{1}{16\pi^3}\int_{x,y} e^{ik(x-y)}\partial_x^2\partial_y^2 \sum_\lambda \epsilon_\lambda^\mu \epsilon_\lambda^\nu A_\mu(x)A_\nu(y)$$

- Repeat the steps to perform a Monte Carlo average

Fast partons vs slow partons

What is beyond MV model?

- Fast partons inside hadron suffer time dilation. — This is only if these partons have a large enough longitudinal momentum. The partons that are too slow must be treated in terms of the usual gauge fields.

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J^{\mu} A_{\mu}$$

- Fast degrees of freedoms are separated from slow Degrees of freedoms (small-x gluons) by a separation scale.

- Partons with $k^+ > \Lambda^+$: static sources ← Large momentum gluons
- Partons with $k^+ < \Lambda^+$: standard quantum fields ← Small-x gluons

- The cut off Λ^+ that separate the fast and the slow partons is arbitrary, and observable quantities should not depend on it.

The change in the physics with the change in the separation scale is governed by **JIMWLK equation.**

JIMWLK equation: Non linear equation takes into account with high density effects that slow down the growth of number of gluons with decreasing x.

$$\frac{\partial \mathscr{W}_{\Lambda^+}}{\partial \ln \Lambda^+} = \mathscr{H} \mathscr{W}_{\Lambda^+}$$

JIMWLK Hamiltonian

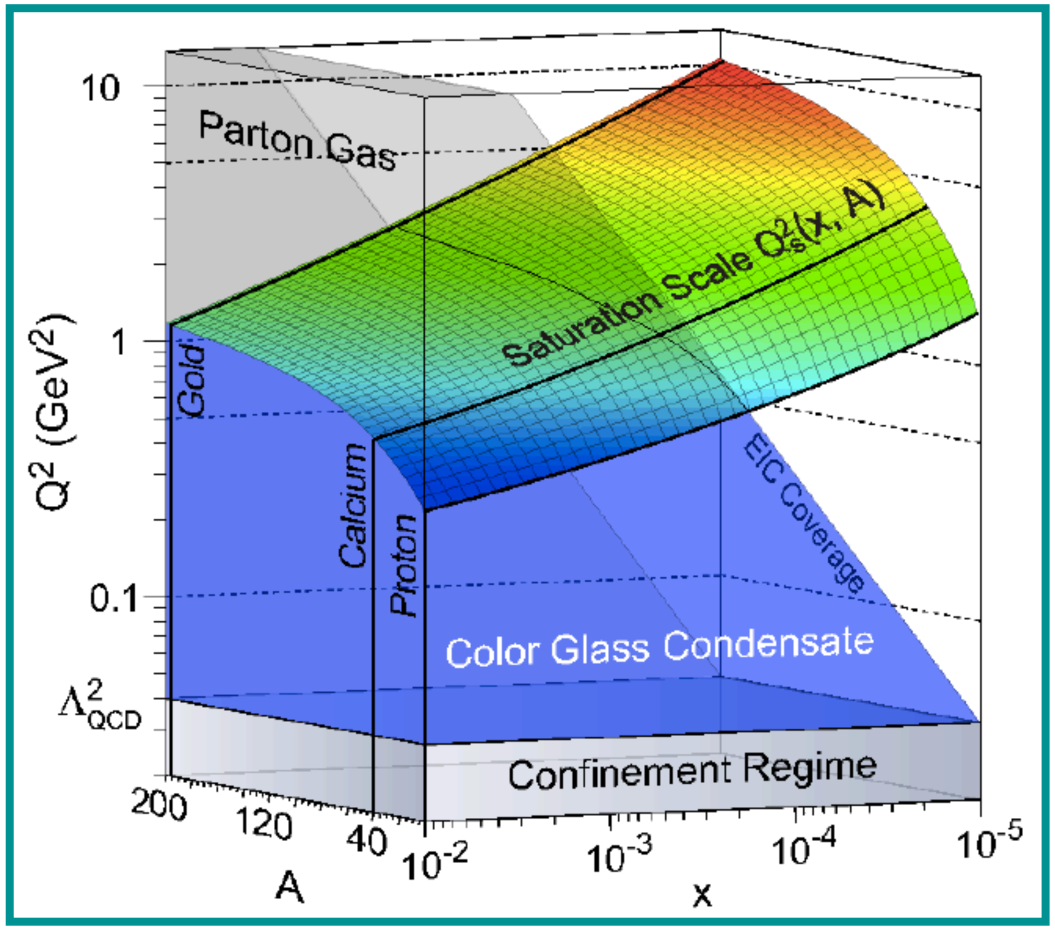
- JIMWLK equation does not predict by itself what the distribution would be.

$$\langle \mathscr{O} \rangle = \frac{\int \mathscr{D}[\rho] \mathscr{W}[\rho] \mathscr{O}[\rho]}{\int \mathscr{D}[\rho] \mathscr{W}[\rho]}$$

- JIMWLK equation predicts how the distribution would change as we lower the cutoff.

Jalilian-Marian - Iancu - McLerran - Weigert - Leonidov - Kovner (1997-2001)

- McLerran Venugopalan model is a plausible initial condition for a large nucleus at large cut off.



Saturation Scale: Saturation of gluon densities at high energy is characterised by the saturation momentum Qs.

$$Q_s^2 \sim A^{1/3}/x^{0.3}$$

- This is a dynamically generated and energy dependent scale below which stochastic independent random multiple scattering no longer valid and non-linear gluon interactions dominates the phase space.

Gluon TMDs at small x   JIMWLK evolution

Jalilian-Marian - Iancu - McLerran
Weigert - Leonidov - Kovner (1997-2001)

- The functional differential evolution equation for is the JIMWLK equation

$$\frac{\partial}{\partial \ln(1/x)} \mathscr{W}_x[\rho] = \mathscr{H}\, \mathscr{W}_x[\rho]$$

$$\mathscr{H} \equiv \frac{1}{2}\int_{xy} \frac{\delta}{\delta\rho_Y^a(x)} \chi^{ab}(x,y) \frac{\delta}{\delta\rho_Y^b(y)}$$ (JIMWLK Hamiltonian)

$$\frac{\delta}{\delta\rho_\tau^a(y)}\tilde{U}(x) = -ig\delta^{(2)}(x-y)\tilde{U}_x T^a$$

$$\frac{\delta}{\delta\rho_\tau^a(y)}\tilde{U}^\dagger(x) = ig\delta^{(2)}(x-y)T^a\tilde{U}_x^\dagger$$

$$\chi^{ab}(x,y) = \frac{1}{\pi}\int \frac{d^2z}{(2\pi)^2}\, \mathscr{K}(x,y,z)\left(1-\tilde{U}_x^\dagger\tilde{U}_z\right)^{fa}\left(1-\tilde{U}_z^\dagger\tilde{U}_y\right)^{fb}$$

$$\mathscr{K}(x,y,z) \equiv \frac{(x-z).(y-z)}{(x-z)^2(z-y)^2}$$

Action of functional derivative on Wilson lines

Within the leading logarithmic accuracy, the CGC effective theory prescribes following energy evolution for general gauge invariant operator

$$\frac{\partial}{\partial Y}\langle\hat{\mathscr{O}}\rangle_Y = \langle\mathscr{H}\hat{\mathscr{O}}\rangle_Y \,. \qquad \mathscr{H} \equiv -\frac{1}{16\pi^3}\int_z \mathscr{M}_{xyz}\left(1+\tilde{U}_x^\dagger\tilde{U}_y-\tilde{U}_x^\dagger\tilde{U}_z-\tilde{U}_z^\dagger\tilde{U}_y\right)^{ab}\frac{\delta}{\delta\rho_x^a}\frac{\delta}{\delta\rho_y^b}$$

Color dipole gluon distribution

Weizsäcker-Williams gluon distribution

Gluon TMDs at small x   JIMWLK evolution

In Muller's dipole model color dipole contains two Wilson lines in their fundamental representation

$$\mathcal{O}^{(2)} = S(x_1, x_2) \equiv \frac{1}{N_c} \mathrm{Tr}\left[U(x_1)U^\dagger(x_2)\right] .$$

The evolution equation for the dipole leads to Balitsky-Kovchegov equation in the large-Nc limit

$$\frac{\partial}{\partial Y} S(x_1, x_2) = \frac{\bar{\alpha}_s}{4\pi} \int_z \frac{(x_1 - x_2)^2}{(x_1 - z)^2 (z - x_2)^2} \left[S(x_1, z)S(z, x_2) - S(x_1, x_2)\right]$$

Next higher point correlators

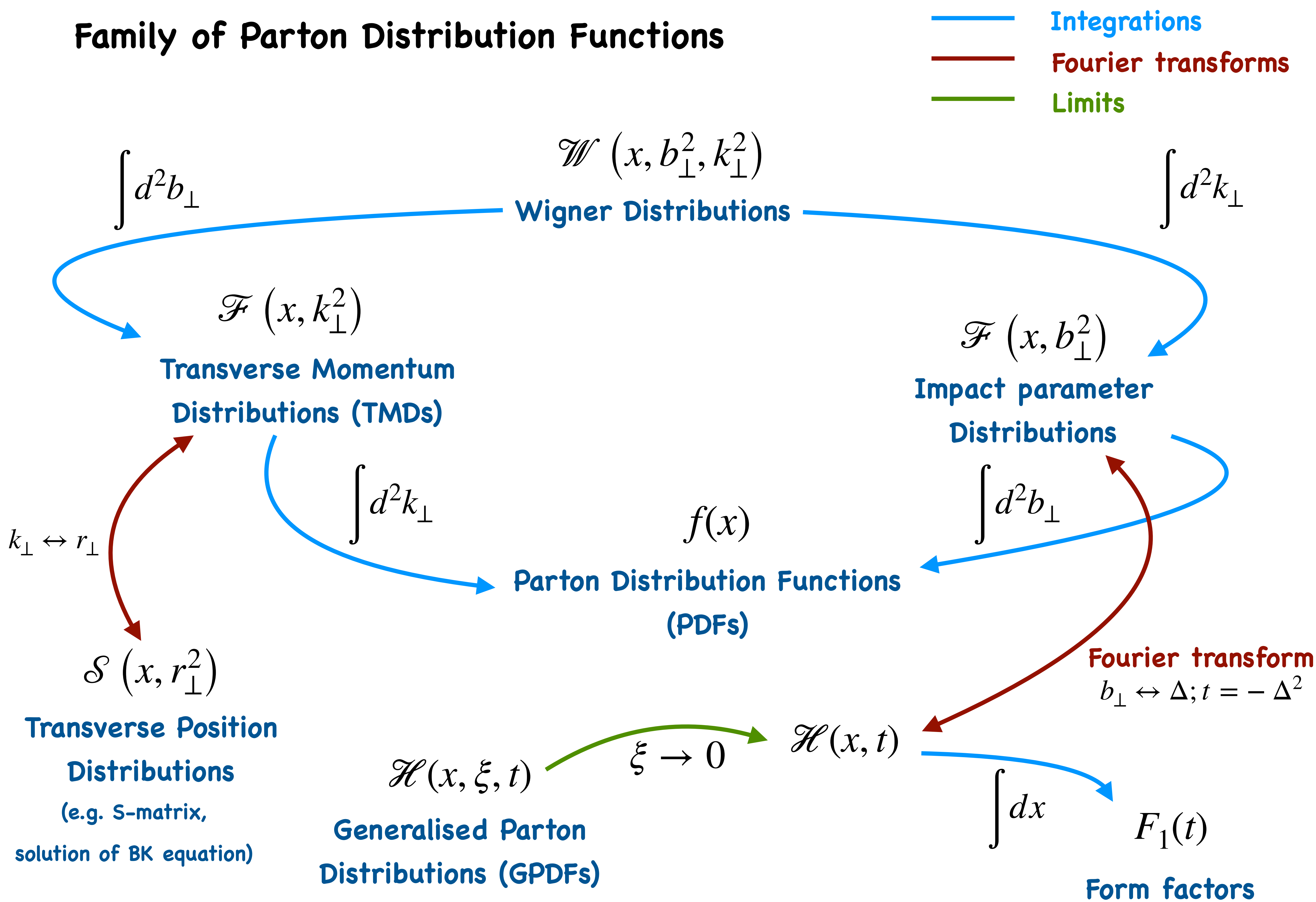
$$\mathcal{S}^{(4)} \equiv \frac{1}{N_c} \mathrm{Tr}\left[U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)\right]$$ → Evolution equation of color quadruple

$$\mathcal{S}^{(6)} \equiv \frac{1}{N_c} \mathrm{Tr}\left[U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)U(x_5)U^\dagger(x_6)\right]$$ → Evolution equation of color sextuple

$$\mathcal{S}^{(2n)} \equiv \frac{1}{N_c} \mathrm{Tr}\left[U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4) \ldots U(x_{2n-1})U^\dagger(x_{2n})\right]$$ → Evolution equation of 2n tuple

"Small-x evolution of 2n-tuple Wilson line correlator revisited: The nonsingular kernels",
Katiza Banu, Mariyah Siddiqah, and Raktim Abir,
Physical Review D 99 (2019), 094017.

Family of Parton Distribution Functions



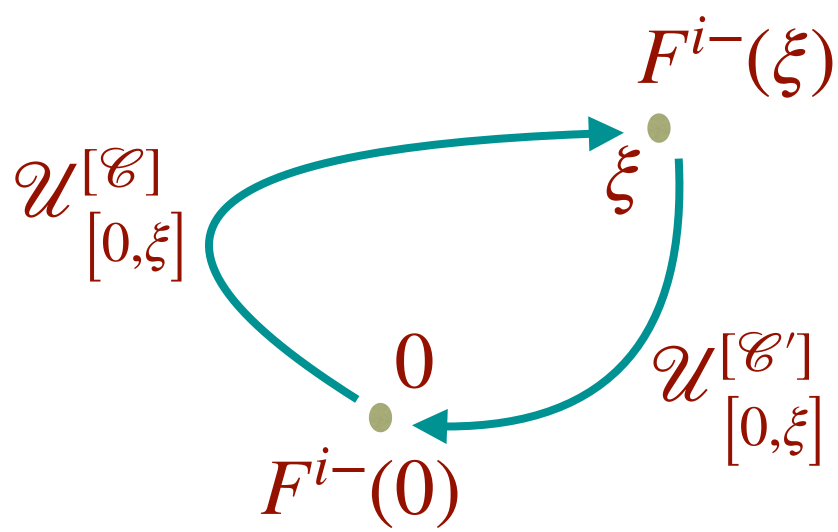
Gluon TMDs at small x   Operator definition

Unpolarised Gluon TMDs

Fourier transform of forward matrix elements of product of two gluon field strength tensors at two space time points

$$\mathscr{F}(x,k_\perp) = 2\int \frac{d\xi^+ d^2\xi_\perp}{(2\pi)^3 P_A^-} e^{ixP_A^-\xi^+ - ik_\perp \xi_\perp} \langle P_A \,|\, Tr\left[F^{i-}(0)\mathscr{U}^{[\mathscr{C}]}_{[0,\xi]} F^{i-}(\xi)\mathscr{U}^{[\mathscr{C}]}_{[\xi,0]}\right] |\, P_A\rangle$$

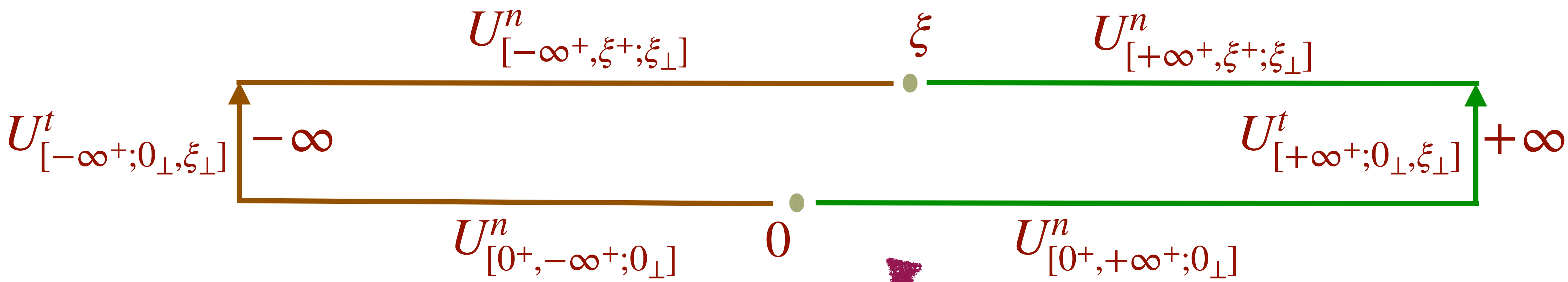
- Gauge links ensure gauge invariant definition of the TMDs.
- Gauge links are path ordered exponentials connecting the field strength tensors along a definite path.
- The path that depends on the actual partonic sub processes.



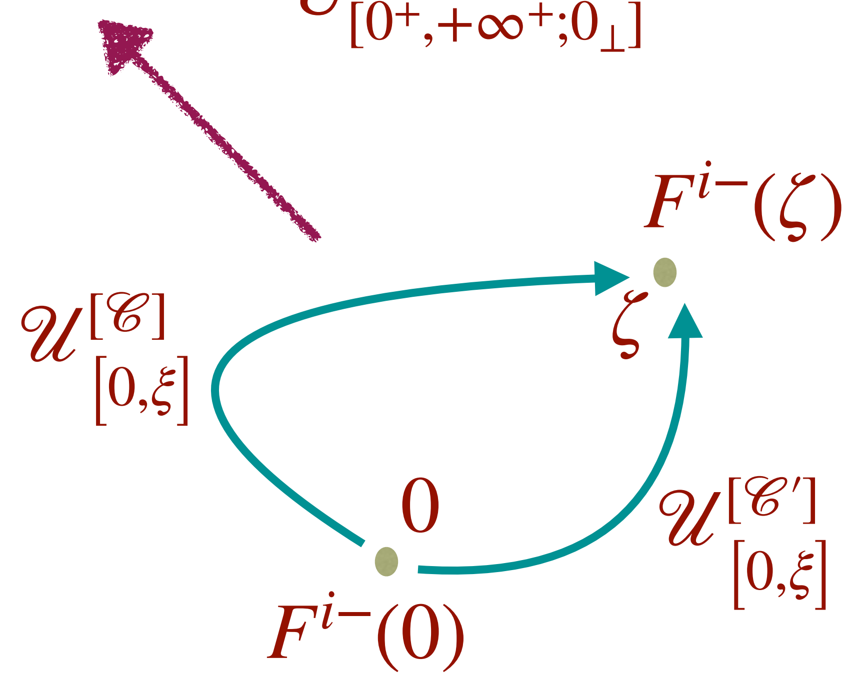
$$\mathscr{U}^{[\mathscr{C}]}_{[0,\xi]} = \mathscr{P}\exp\left[-ig\int_c dz\,.\,A(z)\right]$$

Gluon TMDs at small x   Operator definition

Unpolarised Gluon TMDs



- Gauge links can be constructed by Wilson lines along plus and transverse directions



Future pointing Gauge links $\mathscr{U}^{+}_{[0,\xi]}$

Past pointing Gauge links $\mathscr{U}^{-}_{[0,\xi]}$

Wilson loop $\mathscr{U}^{[\square]}_{[0,\xi]} = \mathscr{U}^{+}_{[0,\xi]}\mathscr{U}^{-\dagger}_{[0,\xi]}$

Future pointing Gauge links

Past pointing Gauge links

Wilson loops

Gluon TMDs at small x   Eight Gluon TMDs

$$\mathcal{F}^{(1)}_{qg}(x_2,k_\perp) \sim \langle P_A | Tr\left[F^{i-}(\xi)\mathcal{U}^{[-]\dagger}F^{i-}(0)\mathcal{U}^{[+]}\right] | P_A\rangle$$

$$\mathcal{F}^{(2)}_{qg}(x_2,k_\perp) \sim \frac{1}{N_c}\langle P_A | Tr\left[F^{i-}(\xi)\mathcal{U}^{[+]\dagger}F^{i-}(0)\mathcal{U}^{[+]}\right] Tr\left[\mathcal{U}^{[\square]}\right] | P_A\rangle$$

$$\mathcal{F}^{(1)}_{gg}(x_2,k_\perp) \sim \frac{1}{N_c}\langle P_A | Tr\left[F^{i-}(\xi)\mathcal{U}^{[-]\dagger}F^{i-}(0)\mathcal{U}^{[+]}\right] Tr\left[\mathcal{U}^{[\square]\dagger}\right] | P_A\rangle$$

$$\mathcal{F}^{(2)}_{gg}(x_2,k_\perp) \sim \frac{1}{N_c}\langle P_A | Tr\left[F^{i-}(\xi)\mathcal{U}^{[\square]\dagger}\right] Tr\left[F^{i-}(0)\mathcal{U}^{[\square]}\right] | P_A\rangle$$

$$\mathcal{F}^{(3)}_{gg}(x_2,k_\perp) \sim \langle P_A | Tr\left[F^{i-}(\xi)\mathcal{U}^{[+]\dagger}F^{i-}(0)\mathcal{U}^{[+]}\right] | P_A\rangle$$

$$\mathcal{F}^{(4)}_{gg}(x_2,k_\perp) \sim \frac{1}{N_c}\langle P_A | Tr\left[F^{i-}(\xi)\mathcal{U}^{[-]\dagger}F^{i-}(0)\mathcal{U}^{[-]}\right] | P_A\rangle$$

$$\mathcal{F}^{(5)}_{gg}(x_2,k_\perp) \sim \frac{1}{N_c}\langle P_A | Tr\left[F^{i-}(\xi)\mathcal{U}^{[\square]\dagger}\mathcal{U}^{[+]\dagger}F^{i-}(0)\mathcal{U}^{[\square]}\mathcal{U}^{[+]}\right] | P_A\rangle$$

$$\mathcal{F}^{(6)}_{gg}(x_2,k_\perp) \sim \frac{1}{N_c}\langle P_A | Tr\left[F^{i-}(\xi)\mathcal{U}^{[+]\dagger}F^{i-}(0)\mathcal{U}^{[+]}\right] Tr\left[\mathcal{U}^{[\square]}\right] Tr\left[\mathcal{U}^{[\square]\dagger}\right] | P_A\rangle$$

How to move to small x?

Future pointing Gauge links

Past pointing Gauge links

Wilson loops

Gluon TMDs at small x   Eight Gluon TMDs

$$\mathcal{F}_{qg}^{(1)}(x_2,k_\perp) \sim \langle P_A|Tr\left[F^{i-}(\xi)\mathcal{U}^{[-]\dagger}F^{i-}(0)\mathcal{U}^{[+]}\right]|P_A\rangle$$

$$\mathcal{F}_{qg}^{(2)}(x_2,k_\perp) \sim \frac{1}{N_c}\langle P_A|Tr\left[F^{i-}(\xi)\mathcal{U}^{[+]\dagger}F^{i-}(0)\mathcal{U}^{[+]}\right]Tr\left[\mathcal{U}^{[\Box]}\right]|P_A\rangle$$

$$\mathcal{F}_{gg}^{(1)}(x_2,k_\perp) \sim \frac{1}{N_c}\langle P_A|Tr\left[F^{i-}(\xi)\mathcal{U}^{[-]\dagger}F^{i-}(0)\mathcal{U}^{[+]}\right]Tr\left[\mathcal{U}^{[\Box]\dagger}\right]|P_A\rangle$$

$$\mathcal{F}_{gg}^{(2)}(x_2,k_\perp) \sim \frac{1}{N_c}\langle P_A|Tr\left[F^{i-}(\xi)\mathcal{U}^{[\Box]\dagger}\right]Tr\left[F^{i-}(0)\mathcal{U}^{[\Box]}\right]|P_A\rangle$$

$$\mathcal{F}_{gg}^{(3)}(x_2,k_\perp) \sim \langle P_A|Tr\left[F^{i-}(\xi)\mathcal{U}^{[+]\dagger}F^{i-}(0)\mathcal{U}^{[+]}\right]|P_A\rangle$$

$$\mathcal{F}_{gg}^{(4)}(x_2,k_\perp) \sim \frac{1}{N_c}\langle P_A|Tr\left[F^{i-}(\xi)\mathcal{U}^{[-]\dagger}F^{i-}(0)\mathcal{U}^{[-]}\right]|P_A\rangle$$

$$\mathcal{F}_{gg}^{(5)}(x_2,k_\perp) \sim \frac{1}{N_c}\langle P_A|Tr\left[F^{i-}(\xi)\mathcal{U}^{[\Box]\dagger}\mathcal{U}^{[+]\dagger}F^{i-}(0)\mathcal{U}^{[\Box]}\mathcal{U}^{[+]}\right]|P_A\rangle$$

$$\mathcal{F}_{gg}^{(6)}(x_2,k_\perp) \sim \frac{1}{N_c}\langle P_A|Tr\left[F^{i-}(\xi)\mathcal{U}^{[+]\dagger}F^{i-}(0)\mathcal{U}^{[+]}\right]Tr\left[\mathcal{U}^{[\Box]}\right]Tr\left[\mathcal{U}^{[\Box]\dagger}\right]|P_A\rangle$$

Dipole gluon distribution

WW distribution

How to move to small x?

Gluon TMDs at small x   How to move to small x? Step 1

Future pointing Gauge links

Past pointing Gauge links

Wilson loops

→ small x

$$\partial_i \mathcal{U}_y = ig \int_{-\infty}^{+\infty} dy^+ \; \mathcal{U}[-\infty, y^+; y_\perp] \; F^{i-}(y) \; \mathcal{U}[y^+, +\infty; y_\perp]$$

Derivatives of Wilson lines

WW distribution

$$\mathcal{F}_{qg}^{(1)}(x_2, k_\perp) = 2 \int \frac{d\xi^+ d^2\xi_\perp}{(2\pi)^3 P_A^-} \; e^{ix_2 P_A^- \xi^+ - ik_\perp . \xi_\perp} \langle P_A | \, Tr \left[F^{i-}(\xi) \mathcal{U}^{[+]\dagger} F^{i-}(0) \mathcal{U}^{[+]} \right] | P_A \rangle$$

Small x ↓

$$\sim -\frac{4}{g^2} \int \frac{d^2x_\perp d^2y_\perp}{(2\pi)^3} \; e^{-ik_\perp (x_\perp - y_\perp)} \; \langle Tr \left[\left(\partial_x U(x_\perp) \right) U(y_\perp) \left(\partial_y U^\dagger(y_\perp) \right) U(x_\perp)^\dagger \right] \rangle_x$$

Dipole gluon distribution

$$\mathcal{F}_{qg}^{(1)}(x_2, k_\perp) = 2 \int \frac{d\xi^+ d^2\xi_\perp}{(2\pi)^3 P_A^-} \; e^{ix_2 P_A^- \xi^+ - ik_\perp . \xi_\perp} \langle P_A | \, Tr \left[F^{i-}(\xi) \mathcal{U}^{[-]\dagger} F^{i-}(0) \mathcal{U}^{[+]} \right] | P_A \rangle$$

Small x ↓

$$\sim \frac{4}{g^2} \int \frac{d^2x_\perp d^2y_\perp}{(2\pi)^3} \; e^{-ik_\perp (x_\perp - y_\perp)} \; \langle Tr \left[\partial_x U(x_\perp) \partial_y U^\dagger(y_\perp) \right] \rangle_x$$

Gluon TMDs at small x How to move to small x? Step 2

Matrix elements average over hadronic state at P →(Small x) Matrix elements average over color glass condensate at some small x

WW distribution

$$\mathcal{F}_{qg}^{(1)}(x_2,k_\perp) = 2\int \frac{d\xi^+ d^2\xi_\perp}{(2\pi)^3 P_A^-}\, e^{ix_2P_A^-\xi^+ - ik_\perp.\xi_\perp}\langle P_A|\, Tr\left[F^{i-}(\xi)\mathcal{U}^{[+]\dagger}F^{i-}(0)\mathcal{U}^{[+]}\right]|P_A\rangle$$

Small x ↓

$$\sim -\frac{4}{g^2}\int \frac{d^2x_\perp d^2y_\perp}{(2\pi)^3}\, e^{-ik_\perp(x_\perp - y_\perp)}\, \langle Tr\left[\left(\partial_x U(x_\perp)\right) U(y_\perp)\left(\partial_y U^\dagger(y_\perp)\right) U(x_\perp)^\dagger\right]\rangle_x$$

color quadruple

Dipole gluon distribution

$$\mathcal{F}_{qg}^{(1)}(x_2,k_\perp) = 2\int \frac{d\xi^+ d^2\xi_\perp}{(2\pi)^3 P_A^-}\, e^{ix_2P_A^-\xi^+ - ik_\perp.\xi_\perp}\langle P_A|\, Tr\left[F^{i-}(\xi)\mathcal{U}^{[-]\dagger}F^{i-}(0)\mathcal{U}^{[+]}\right]|P_A\rangle$$

Small x ↓

$$\sim \frac{4}{g^2}\int \frac{d^2x_\perp d^2y_\perp}{(2\pi)^3}\, e^{-ik_\perp(x_\perp - y_\perp)}\, \langle Tr\left[\partial_x U(x_\perp)\partial_y U^\dagger(y_\perp)\right]\rangle_x$$

color dipole

Estimate CGC / JIMWLK

Gluon TMDs at small x

WW distribution

$$\mathscr{F}_{qg}^{(1)}(x_2,k_\perp) = 2\int \frac{d\xi^+ d^2\xi_\perp}{(2\pi)^3 P_A^-} e^{ix_2P_A^-\xi^+ - ik_\perp.\xi_\perp}\langle P_A|Tr\left[F^{i-}(\xi)\mathscr{U}^{[+]\dagger}F^{i-}(0)\mathscr{U}^{[+]}\right]|P_A\rangle$$

Small x ↓

$$\sim -\frac{4}{g^2}\int\frac{d^2x_\perp d^2y_\perp}{(2\pi)^3} e^{-ik_\perp(x_\perp - y_\perp)}\langle Tr\left[\left(\partial_x U(x_\perp)\right) U(y_\perp)\left(\partial_y U^\dagger(y_\perp)\right) U(x_\perp)^\dagger\right]\rangle_x$$

color quadruple

Dipole gluon distribution

$$\mathscr{F}_{qg}^{(1)}(x_2,k_\perp) = 2\int \frac{d\xi^+ d^2\xi_\perp}{(2\pi)^3 P_A^-} e^{ix_2P_A^-\xi^+ - ik_\perp.\xi_\perp}\langle P_A|Tr\left[F^{i-}(\xi)\mathscr{U}^{[-]\dagger}F^{i-}(0)\mathscr{U}^{[+]}\right]|P_A\rangle$$

Small x ↓

$$\sim \frac{4}{g^2}\int\frac{d^2x_\perp d^2y_\perp}{(2\pi)^3} e^{-ik_\perp(x_\perp - y_\perp)}\langle Tr\left[\partial_x U(x_\perp)\partial_y U^\dagger(y_\perp)\right]\rangle_x$$

color dipole

We take a special solution of BK equation or S-matrix (Levin-Tuchin solution) that is valid for small-x and large transverse separation (large) to derive Color dipole distribution that would be valid for small-x and small transverse momentum (small x).

Siddiqah, Vasim, Banu, Bhattacharyya, Abir,
PRD 97 (2018) 054009.

Gluon TMDs at small x

TMD (color dipole distribution) at small-x and small transverse momentum

$$x\mathscr{F}^{DP}(x,k_\perp) = \frac{S_\perp N_c}{2\pi^2\alpha_s} k_\perp^2 \int \frac{d^2 r_\perp}{(2\pi)^2} \; e^{-ik_\perp . r_\perp} \, S(x,r_\perp)$$

Small $k_\perp \leftrightarrow$ Large $r_\perp$

Fourier Transform

$$S(x,r_\perp) = S_0 \exp\left(-\tau \ln^2\left[r_\perp^2 Q_s^2(Y)\right]\right)$$

(Levin-Tuchin solution) valid for small-x and large transverse separation (large $r_\perp$)

The form is not gaussian but gaussian in logarithm!

We developed new mathematical techniques to Fourier transform lognormal distributions in two dimension

Siddiqah, Vasim, Banu, Bhattacharyya, Abir, PRD 97 (2018) 054009.

Bell Polynomials - Combinatorial meaning

- Bell polynomials appear in the study of set partition.

Examples

$\{A, B, C\}$

$\{\, - , - \,\}\ \ \{\, - \,\}$ 3 ways

$\{A, B\}\ \ \{C\}$

$\{B, C\}\ \ \{A\}$

$\{C, A\}\ \ \{B\}$

$$B_{3,2}\left(x_1, x_2\right) = 3x_2x_1$$

Partial Bell Polynomials

$\{A, B, C, D, E, F\}$

$\{\, - , - , - , - \,\}\ \ \{\, - \,\}\ \ \{\, - \,\}$ 15 ways

$\{\, - , - , - \,\}\ \ \{\, - , - \,\}\ \ \{\, - \,\}$ 60 ways

$\{\, - , - \,\}\ \ \{\, - , - \,\}\ \ \{\, - , - \,\}$ 15 ways

$$B_{6,3}\left(x_1, x_2, x_3, x_4\right) = 15x_4x_1^2 + 60x_3x_2x_1 + 15x_2^3$$

Partial Bell Polynomials

- Bell polynomials appear in the study of product of several series or when series exponentiate.

Examples

$$\frac{1}{k!}\left(\sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1,\ldots,x_{n-k+1})\frac{t^n}{n!}$$

$$\exp\left(u\sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right) = \sum_{n\geq k\geq 0}^{\infty} B_{n,k}(x_1,\ldots,x_{n-k+1})\frac{t^n}{n!}u^k$$

$$\exp\left(\sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right) = \sum_{n=0}^{\infty} B_{n}(x_1,\ldots,x_{n})\frac{t^n}{n!}$$

$$B_n\left(x_1,x_2,\ldots,x_3\right) = \sum_{k=1}^{n} B_{n,k}\left(x_1,x_2,\ldots,x_{n-k+1}\right)$$

Complete Bell Polynomials Partial Bell Polynomials

- Bell polynomials appear in the study of product of several series or when series exponentiate.

Examples

$$\mathcal{F} = \frac{S_\perp}{2\pi^2}\frac{N_c}{\alpha_s}k_\perp^2\frac{d}{dk_\perp^2}\sum_0^\infty\frac{(-\tau)^n}{n!}B_{2n}\left[\ln\frac{k_\perp^2}{4Q_s^2}+2\gamma,\ 0,\ 2!2\zeta(3),\ 0,\ 4!2\zeta(5),\ \ldots,\ (2n-2)!2\zeta(2n-1),\ 0\right]$$

Leading-log approximation

$$\mathcal{F} = \frac{S_\perp}{2\pi^2}\frac{N_c}{\alpha_s}k_\perp^2\frac{d}{dk_\perp^2}\sum_0^\infty\frac{(-\tau)^n}{n!}\left(\ln\frac{k_\perp^2}{4Q_s^2}+2\gamma\right)^{2n}$$

$$x\mathcal{F}(x,k_\perp)\big|_{Q_s>k_\perp\gg\lambda_{QCD}} \approx -\frac{S_\perp N_c\tau}{\pi^3\alpha_s}\ln\left(\frac{k_\perp^2}{4Q_s^2(Y)}\right)\exp\left[-\tau\ln^2\left(\frac{k_\perp^2}{4Q_s^2(Y)}\right)\right]$$

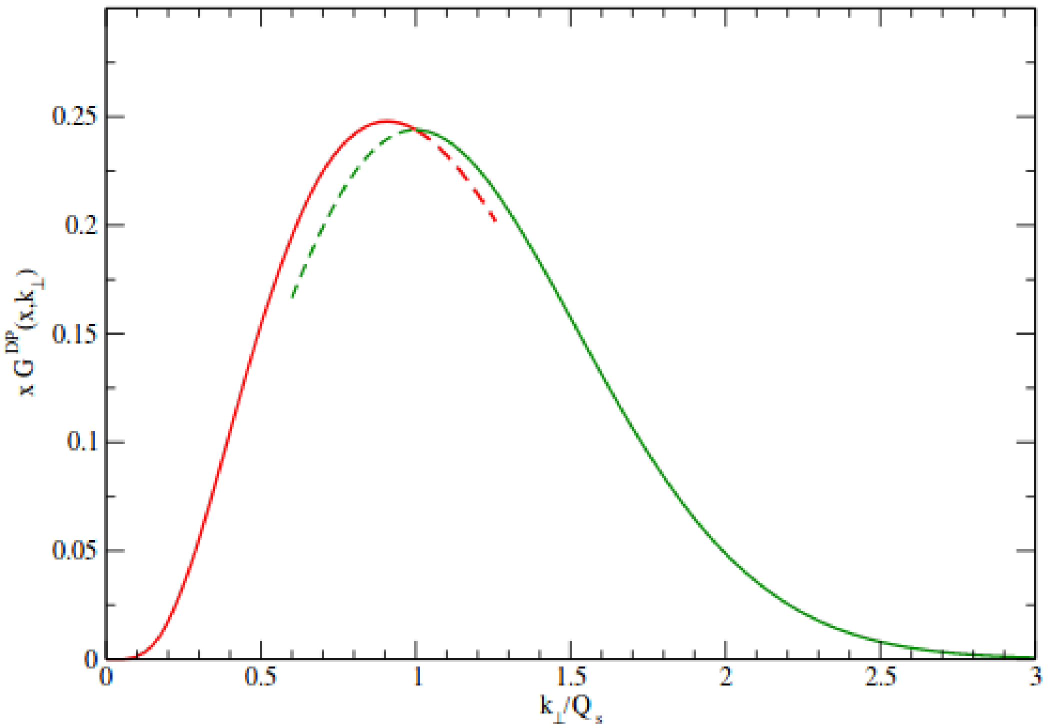
Gluon TMDs at small x

TMD (color dipole distribution) at small-x and small transverse momentum

$$xG^{DP}(x,k_\perp)|_{Q_s>k_\perp\gg\lambda_{QCD}} \approx -\frac{S_\perp N_c \tau}{\pi^3\alpha_s}\ln\left(\frac{k_\perp^2}{4Q_s^2(Y)}\right)\exp\left[-\tau\ln^2\left(\frac{k_\perp^2}{4Q_s^2(Y)}\right)\right]$$

When resumming the series in leading log accuracy, the results showing up striking similarity with the Sudakov form factor.

Inside the saturation region, the dipole gluon distribution is expected to go to zero in zero momentum.



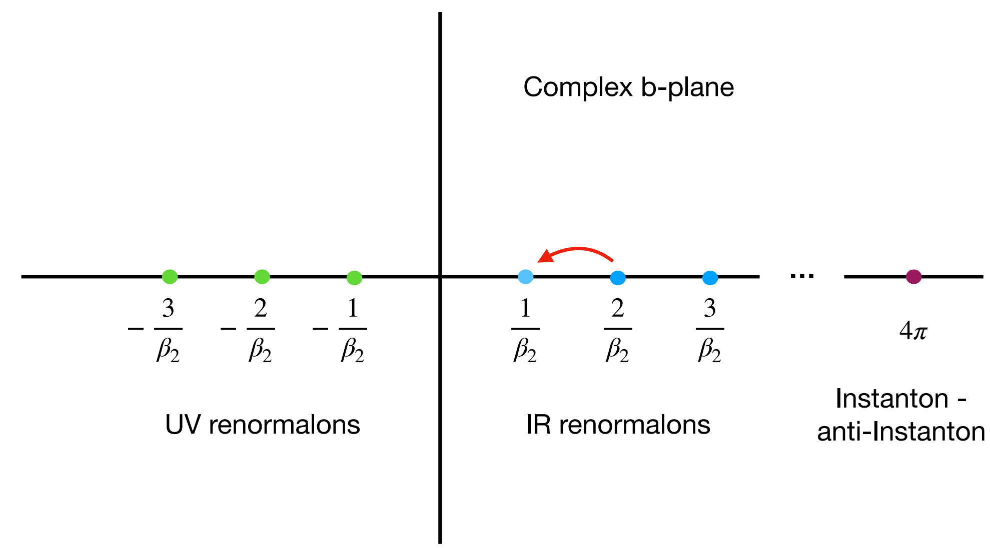
Siddiqah, Vasim, Banu, Bhattacharyya, Abir,
PRD 97 (2018) 054009.

Gluon TMDs at small x

Infrared renormalons

TMD (color dipole distribution) at small-x and small transverse momentum

$$xG^{DP}(x,k_\perp)|_{Q_s>k_\perp\gg\lambda_{QCD}} \approx -\frac{S_\perp N_c \tau}{\pi^3 \alpha_s} \ln\left(\frac{k_\perp^2}{4Q_s^2(Y)}\right) \exp\left[-\tau \ln^2\left(\frac{k_\perp^2}{4Q_s^2(Y)}\right)\right]$$



The uncertainties from the infrared renormalons in the (color dipole) gluon distribution is estimate

It is shown that non-linear saturation effects at small-x shift the first IR pole at the Borel plane from $2/\beta_2$ to $1/\beta_2$.

Vasim, Abir,
Nucl. Phys. B 953 (2020) 114961

Gluon TMDs at small x

Solution to BK equation

$$\frac{\partial S(x_{10};Y)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2}\int d^2z \frac{x_{10}^2}{x_{20}^2 x_{21}^2}\left[S(x_{20};Y)S(x_{21};Y) - S(x_{10};Y)\right]$$

$$S(x_{10}, y=0) = \exp\left[-\left(\frac{r^2 Q_{s0}^2}{4}\right)\ln\left(\frac{1}{x_{10}\Lambda_{QCD}}\right)\right]$$

Initial condition (McLerran Venugopalan model)

$$S(x_{10}, Y) = \exp\left(-\tau \ \ln^2\left[x_{10}^2 Q_s^2(Y)\right]\right)$$

Solution Deep inside the saturation region $(x_{10}Q_s \gg 1)$

(Levin-Tuchin solution)

$$S(r_\perp, Y) = \exp\left(-1.48 \ x_{10}^2 Q_s^2(Y)\right)$$

Solution outside Saturation region $(x_{10}Q_s \ll 1)$

(McLerran Venugopalan model)

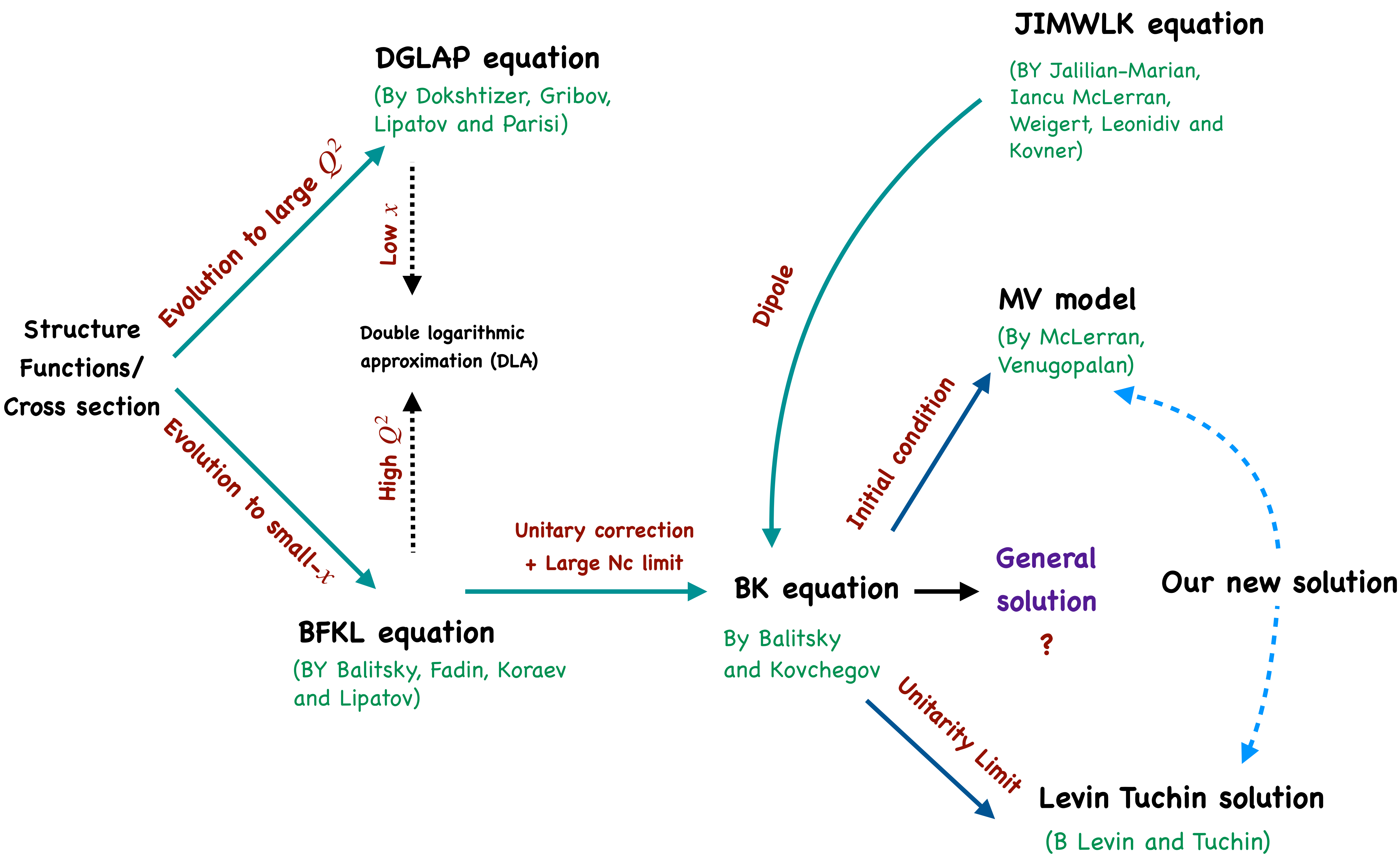
- To derive the new solution we adopted a dipole transverse-width-dependent cutoff in order to regulate the dipole integral.

$$S(x_\perp, Y) = \exp\left(\frac{1+2i\nu_0}{2\chi(0,\nu_0)}\ \mathrm{Li}_2\left[-\lambda_1 x_{10}^2 Q_s^2(Y)\right]\right)$$

- We also have taken care of all the higher-order terms (higher order in the cutoff) that have been reasonably neglected before.

Siddiqah and Abir,
PRD 95 (2017), 074035

QCD evolution equations



JIMWLK equation for 2n tuple

$$
\frac{\partial}{\partial Y}\mathrm{Tr}\left[U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4)\ldots U(x_{2n-1})U^\dagger(x_{2n})\right] = \frac{\bar{\alpha}_s}{4\pi}\left(\frac{1}{1+\delta_{n,1}}\right)
$$

$$
\int_z \sum_{k=0}^{\lfloor n/2\rfloor-1}\sum_{l=0}^{n-1}\mathscr{K}^{(2l;2l+2k+2)}_{(2l+1;2l+2k+1)}\mathrm{Tr}\left[U(x_{2l+1})U^\dagger(x_{2l+2})\ldots U(x_{2l+1+2k})U^\dagger(z)\right]\mathrm{Tr}\left[U(z)U^\dagger(x_{2l+2k+2})\ldots U(x_{2l-1})U^\dagger(x_{2l})\right]
$$

$$
+\sum_{k=0}^{\lfloor n/2\rfloor-1}\sum_{l=0}^{n-1}\mathscr{K}^{(2l+1;2l+2k+3)}_{(2l+2;2l+2k+2)}\mathrm{Tr}\left[U^\dagger(x_{2l+2})U(x_{2l+3})\ldots U^\dagger(x_{2l+2+2k})U(z)\right]\mathrm{Tr}\left[U^\dagger(z)U(x_{2l+2k+3})\ldots U^\dagger(x_{2l})U(x_{2l+1})\right]
$$

$$
+\sum_{k=0}^{\lceil n/2\rceil-2}\sum_{l=0}^{n-1}\mathscr{K}^{(2l;2l+2k+3)}_{(2l+1;2l+2k+2)}\mathrm{Tr}\left[U(x_{2l+1})U^\dagger(x_{2l+2})\ldots U^\dagger(x_{2l+1+2k+1})\right]\mathrm{Tr}\left[U(x_{2l+2k+3})U^\dagger(x_{2l+2k+4})\ldots\ldots U^\dagger(x_{2l})\right]
$$

$$
+\sum_{k=0}^{\lceil n/2\rceil-2}\sum_{l=0}^{n-1}\mathscr{K}^{(2l+1;2l+2k+4)}_{(2l+2;2l+2k+3)}\mathrm{Tr}\left[U^\dagger(x_{2l+2})U(x_{2l+3})\ldots U(x_{2l+2k+3})\right]\mathrm{Tr}\left[U^\dagger(x_{2l+2k+4})U(x_{2l+2k+5})\ldots U(x_{2l+1})\right]
$$

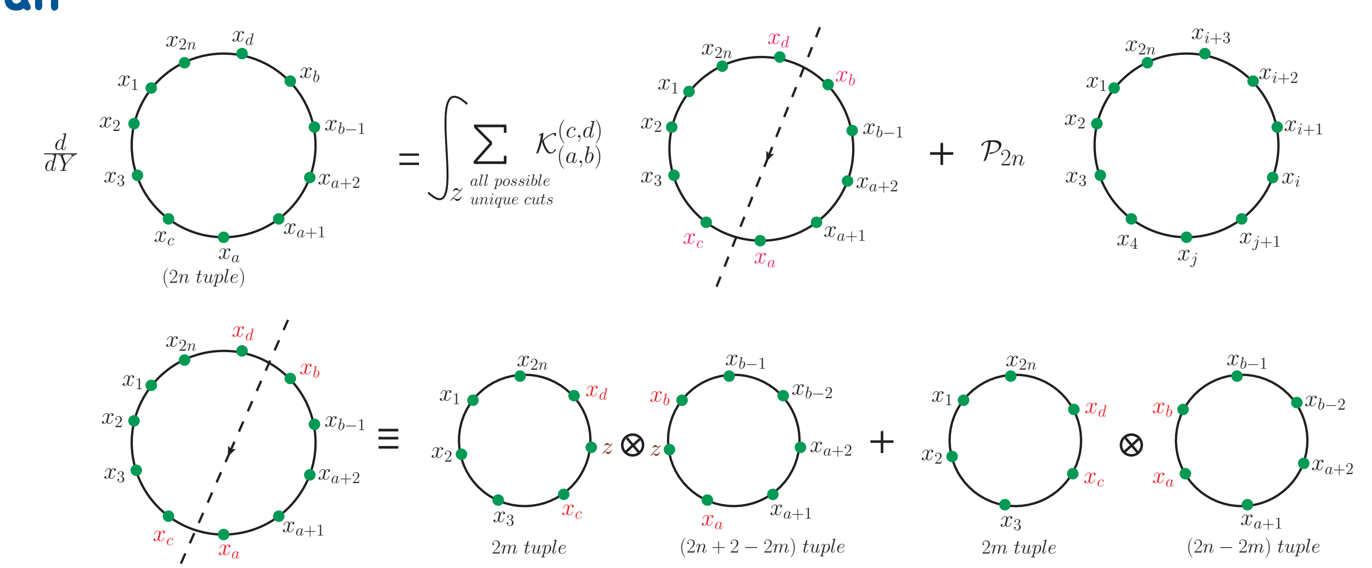
$$
+\delta_{1,n\bmod 2}\sum_{l=0}^{\lceil n/2\rceil-1}\mathscr{K}^{(2l;2l+n+1)}_{(2l+1;2l+n)}\mathrm{Tr}\left[U(x_{2l+1})U^\dagger(x_{2l+2})\ldots U(x_{2l+n})U^\dagger(z)\right]\mathrm{Tr}\left[U(z)U^\dagger(x_{2l+n+1})\ldots U(x_{2l-1})U^\dagger(x_{2l})\right]
$$

$$
+\delta_{1,n\bmod 2}\sum_{l=0}^{\lceil n/2\rceil-2}\mathscr{K}^{(2l+1;2l+n+2)}_{(2l+2;2l+n+1)}\mathrm{Tr}\left[U^\dagger(x_{2l+2})U(x_{2l+3})\ldots U^\dagger(x_{2l+n+1})U(z)\right]\mathrm{Tr}\left[U^\dagger(z)U(x_{2l+n+2})\ldots U^\dagger(x_{2l})U(x_{2l+1})\right]
$$

In this work we present small-x evolution equation for general 2n-tuple Wilson line correlator.

It is an integro-dierential equation where all real and virtual terms are explicit.

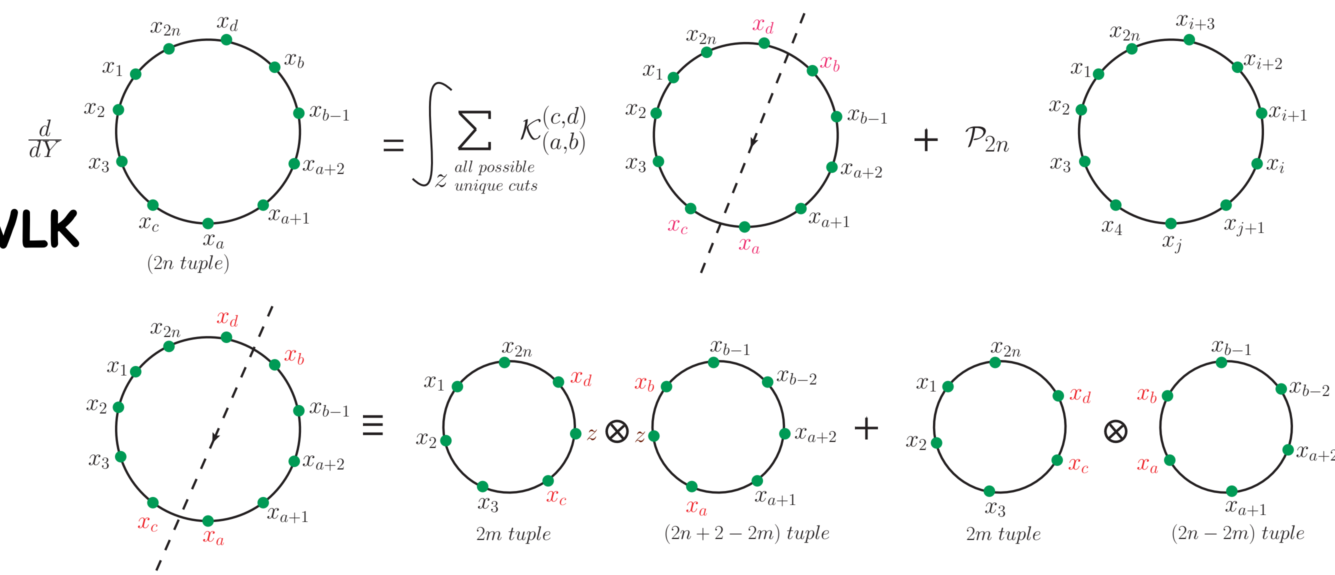
This is done by operating JIMWLK Hamiltonian on a single trace of 2n-Wilson lines in their fundamental representation.



Schematic representation of our equation

Banu, Siddiqah, Abir,
PRD 99 (2019), 094017.

JIMWLK equation for 2n tuple



Equivalence of Balitsky hierarchy and JIMWLK

Our study conform the equivalence of Balitsky hierarchy of evolution and the JIMWLK evolution for this general 2n-tuple color correlator..

Schematic representation of our equation

Non-singular kernels

Kernels is SL(2,Z) invariant.

$$\mathscr{K}^{(c;d)}_{(a;b)} = \frac{(x_a - x_d)^2}{(x_a - z)^2(z - x_d)^2} + \frac{(x_b - x_c)^2}{(x_b - z)^2(z - x_c)^2} - \frac{(x_a - x_b)^2}{(x_a - z)^2(z - x_b)^2} - \frac{(x_c - x_d)^2}{(x_c - z)^2(z - x_d)^2}$$

$$\mathscr{P}_{2n} = \sum_{j=1}^{2n} \frac{(x_j - x_{j+1})^2}{(x_j - z)^2(z - x_{j+1})^2}$$

Banu, Siddiqah, Abir,
PRD 99 (2019), 094017.

Gluon TMDs at small x

Small x evolution of jet quenching parameter

Jet quenching parameter

$$\hat{q} = \frac{\langle k_\perp^2 \rangle}{L} = \frac{1}{4\pi L}\int dk_\perp^2 \, k_\perp^2 \, P\left(k_\perp^2\right)$$

Probability of transverse deflection

Higher twist formalism of jet quenching

$$\hat{q} \sim \langle P | \mathscr{F}^{i+}(0)\mathscr{F}^{i+}(y^-) | P \rangle$$

Can be expressed as the Fourier transform of two light like path ordered transversely separated Wilson lines.

$$P\left(k_\perp^2\right) = \int d^2 r_\perp \, e^{-ik_\perp r_\perp} \, S\left(r_\perp\right)$$

$$\hat{q} \propto \left(\frac{1}{x}\right)^{0.3\alpha_s}$$

Abir

PLB 748 (2015) 467-471