Quantum Signatures of Spacetime "Graininess"

Sachindeo Vaidya

Centre for High Energy Physics,
Indian Institute of Science, Bangalore

TIFR Theory Colloquium
15th September 2009

Introduction

Introduction

Introduction

Introduction

Introduction

Introduction

A look at distances

$$
\begin{aligned}
\text{Milky Way} &\sim 10^{21}\, m \\
\text{Solar System} &\sim 10^{12}\, m \\
\text{Car} &\sim 1\, m \\
\text{Atom} &\sim 10^{-10}\, m \\
\text{Proton} &\sim 10^{-15}\, m \\
\text{GUT scale} &\sim 10^{-32}\, m \\
\text{Planck scale} &\sim 10^{-35}\, m
\end{aligned}
$$

How do we measure distances?

- For galactic distances, there are (indirect) techniques involving angular size and standard candle, also red-shift data, and so on.
- For planetary distances, one can use Kepler's laws.
- For even smaller distances (cars, shoes, ...), we can use the tape measure.
- For atomic sizes and smaller, we need to use particles whose Compton wavelength is comparable to the size of the object.

How do we measure distances?

- For galactic distances, there are (indirect) techniques involving angular size and standard candle, also red-shift data, and so on.
- For planetary distances, one can use Kepler's laws.
- For even smaller distances (cars, shoes, ...), we can use the tape measure.
- For atomic sizes and smaller, we need to use particles whose Compton wavelength is comparable to the size of the object.

How do we measure distances?

- For galactic distances, there are (indirect) techniques involving angular size and standard candle, also red-shift data, and so on.
- For planetary distances, one can use Kepler's laws.
- For even smaller distances (cars, shoes, ...), we can use the tape measure.
- For atomic sizes and smaller, we need to use particles whose Compton wavelength is comparable to the size of the object.

How do we measure distances?

- For galactic distances, there are (indirect) techniques involving angular size and standard candle, also red-shift data, and so on.
- For planetary distances, one can use Kepler's laws.
- For even smaller distances (cars, shoes, ...), we can use the tape measure.
- For atomic sizes and smaller, we need to use particles whose Compton wavelength is comparable to the size of the object.

(Sub-)Atomic scale measurements

- On the scale of atomic distance and smaller, new effects come into play because of quantum mechanics:
- The position and momentum of a particle cannot be measured simultaneously to infinite accuracy.
- The energy and lifetime of a quantum state (particle) cannot be measured to arbitrary accuracy.

(Sub-)Atomic scale measurements

- On the scale of atomic distance and smaller, new effects come into play because of quantum mechanics:
- The position and momentum of a particle cannot be measured simultaneously to infinite accuracy.
- The energy and lifetime of a quantum state (particle) cannot be measured to arbitrary accuracy.

(Sub-)Atomic scale measurements

- On the scale of atomic distance and smaller, new effects come into play because of quantum mechanics:
- The position and momentum of a particle cannot be measured simultaneously to infinite accuracy.
- The energy and lifetime of a quantum state (particle) cannot be measured to arbitrary accuracy.

Quantum Mechanics and Classical Gravity

- Gravity and quantum mechanics are both important when distances are of the order of the Planck length $\ell_P = \left(\frac{G\hbar}{c^3}\right)^{1/2}$.
- In order to probe physics at the length scale ℓ_P, the Compton wavelength $\hbar/Mc$ of the probe must satisfy

$$\frac{\hbar}{Mc} \lesssim \ell_P \implies M \gtrsim \frac{\hbar}{\ell_P c}$$

Quantum Mechanics and Classical Gravity

- Gravity and quantum mechanics are both important when distances are of the order of the Planck length $\ell_P = \left(\frac{G\hbar}{c^3}\right)^{1/2}$.
- In order to probe physics at the length scale ℓ_P, the Compton wavelength $\hbar/Mc$ of the probe must satisfy

$$\frac{\hbar}{Mc} \lesssim \ell_P \implies M \gtrsim \frac{\hbar}{\ell_P c}$$

Quantum Mechanics and Classical Gravity

- Classical gravity tells us that because of self-gravitation, mass (or energy) concentrated in a region of space can continue to collapse further.
- If this region of the order of the Schwarzschild radius, a black hole can form, and we lose access to the region beyond the black hole horizon.
- In our case, this large mass concentrated in so small a volume (ℓ_P^3) will lead to the formation of black holes and horizons.
- This suggests a fundamental length limiting spatial localization.
- Similar arguments can also be made about time localization.

Quantum Mechanics and Classical Gravity

- Classical gravity tells us that because of self-gravitation, mass (or energy) concentrated in a region of space can continue to collapse further.
- If this region of the order of the Schwarzschild radius, a black hole can form, and we lose access to the region beyond the black hole horizon.
- In our case, this large mass concentrated in so small a volume (ℓ_P^3) will lead to the formation of black holes and horizons.
- This suggests a fundamental length limiting spatial localization.
- Similar arguments can also be made about time localization.

Quantum Mechanics and Classical Gravity

- Classical gravity tells us that because of self-gravitation, mass (or energy) concentrated in a region of space can continue to collapse further.
- If this region of the order of the Schwarzschild radius, a black hole can form, and we lose access to the region beyond the black hole horizon.
- In our case, this large mass concentrated in so small a volume (ℓ_P^3) will lead to the formation of black holes and horizons.
- This suggests a fundamental length limiting spatial localization.
- Similar arguments can also be made about time localization.

Quantum Mechanics and Classical Gravity

- Classical gravity tells us that because of self-gravitation, mass (or energy) concentrated in a region of space can continue to collapse further.
- If this region of the order of the Schwarzschild radius, a black hole can form, and we lose access to the region beyond the black hole horizon.
- In our case, this large mass concentrated in so small a volume (ℓ_P^3) will lead to the formation of black holes and horizons.
- This suggests a fundamental length limiting spatial localization.
- Similar arguments can also be made about time localization.

Quantum Mechanics and Classical Gravity

- Classical gravity tells us that because of self-gravitation, mass (or energy) concentrated in a region of space can continue to collapse further.
- If this region of the order of the Schwarzschild radius, a black hole can form, and we lose access to the region beyond the black hole horizon.
- In our case, this large mass concentrated in so small a volume (ℓ_P^3) will lead to the formation of black holes and horizons.
- This suggests a fundamental length limiting spatial localization.
- Similar arguments can also be made about time localization.

Spacetime Uncertainty Relations

Doplicher-Fredenhagen-Roberts, 1995

"Attempts to localize with extreme precision cause gravitational collapse, so spacetime below the Planck scale has no operational meaning."

- More precisely, we get the spacetime uncertainties:

$$\Delta x_0 \left(\sum_i \Delta x_i \right) \gtrsim \ell_P^2, \qquad \sum_{1 \leq i < j \leq 3} \Delta x_i \Delta x_j \gtrsim \ell_P^2$$

Spacetime Uncertainty Relations

Doplicher-Fredenhagen-Roberts, 1995

"Attempts to localize with extreme precision cause gravitational collapse, so spacetime below the Planck scale has no operational meaning."

- More precisely, we get the spacetime uncertainties:

$$\Delta x_0 \left(\sum_i \Delta x_i \right) \gtrsim \ell_P^2, \qquad \sum_{1 \le i < j \le 3} \Delta x_i \Delta x_j \gtrsim \ell_P^2$$

Noncommutative Spacetime (Moyal Algebra)

- A concrete model for these uncertainties is the algebra generated by operators $\hat{x}_\mu$:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu},$$

where $\theta_{\mu\nu}$ is a (fixed) constant antisymmetric matrix.

- This model for noncommutative spacetime is a not a model for quantum gravity. Rather, it is a bridge between standard quantum theory and quantum gravity (whatever it might be).

Noncommutative Spacetime (Moyal Algebra)

- A concrete model for these uncertainties is the algebra generated by operators $\hat{x}_\mu$:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu},$$

  where $\theta_{\mu\nu}$ is a (fixed) constant antisymmetric matrix.
- This model for noncommutative spacetime is a not a model for quantum gravity. Rather, it is a bridge between standard quantum theory and quantum gravity (whatever it might be).

Analogy with Quantum Mechanics

- Quantum mechanics emerges because it is operationally meaningless to localize points in classical phase space.
- Classical phase space (a commutative manifold) is replaced in QM by a "noncommutative" manifold $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$.
- This leads to "cells" in phase space, giving us Planck's radiation law, and avoiding the ultra-violet catastrophe of Rayleigh-Jeans law.

Analogy with Quantum Mechanics

- Quantum mechanics emerges because it is operationally meaningless to localize points in classical phase space.
- Classical phase space (a commutative manifold) is replaced in QM by a “noncommutative” manifold $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$.
- This leads to “cells” in phase space, giving us Planck’s radiation law, and avoiding the ultra-violet catastrophe of Rayleigh-Jeans law.

Analogy with Quantum Mechanics

- Quantum mechanics emerges because it is operationally meaningless to localize points in classical phase space.
- Classical phase space (a commutative manifold) is replaced in QM by a "noncommutative" manifold $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$.
- This leads to "cells" in phase space, giving us Planck's radiation law, and avoiding the ultra-violet catastrophe of Rayleigh-Jeans law.

The Moyal Algebra

- Our starting point is the set of commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}.$$

- This algebra has the advantage that it can be realized in terms of ordinary functions on Minkowski space, but with a new noncommutative product:

$$\begin{aligned} f(x) * g(x) &= f(x) e^{\frac{i}{2}\overleftarrow{\partial_\mu}\theta^{\mu\nu}\overrightarrow{\partial_\nu}} g(x) \\ &\simeq f(x)\cdot g(x) + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f(x)\partial_\nu g(x) + \cdots \end{aligned}$$

The Moyal Algebra

- Our starting point is the set of commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}.$$

- This algebra has the advantage that it can be realized in terms of ordinary functions on Minkowski space, but with a new noncommutative product:

$$\begin{aligned} f(x) * g(x) &= f(x) e^{\frac{i}{2}\overleftarrow{\partial_\mu}\theta^{\mu\nu}\overrightarrow{\partial_\nu}} g(x) \\ &\simeq f(x)\cdot g(x) + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f(x)\partial_\nu g(x) + \cdots \end{aligned}$$

How do we do quantum mechanics?

- Our interest is in understanding quantum theory on this noncommutative space.
- Let us look at a two-dimensional example. The fundamental commutation relations are:

$$[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hbar, \quad [\hat{x}, \hat{y}] = i\theta.$$

- We can solve the simplest non-trivial problem: particle in a "circular" well.

How do we do quantum mechanics?

- Our interest is in understanding quantum theory on this noncommutative space.
- Let us look at a two-dimensional example. The fundamental commutation relations are:

$$[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hbar, \quad [\hat{x}, \hat{y}] = i\theta.$$

- We can solve the simplest non-trivial problem: particle in a "circular" well.

How do we do quantum mechanics?

- Our interest is in understanding quantum theory on this noncommutative space.
- Let us look at a two-dimensional example. The fundamental commutation relations are:

$$[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hbar, \quad [\hat{x}, \hat{y}] = i\theta.$$

- We can solve the simplest non-trivial problem: particle in a “circular” well.

How do we do quantum mechanics?

- Our interest is in understanding quantum theory on this noncommutative space.
- Let us look at a two-dimensional example. The fundamental commutation relations are:

$$[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hbar, \quad [\hat{x}, \hat{y}] = i\theta.$$

- We can solve the simplest non-trivial problem: particle in a "circular" well.

Piecewise constant potential

- The Hamiltonian for the circular well problem is

$$H = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m_0} + V(\hat{x}, \hat{y})$$

  where $V(\hat{x}, \hat{y})$ is a "piecewise" constant potential: it is $(-V_0)$ in a circular region of radius R around the origin, and zero elsewhere.

- In the commutative case, the spectrum for the "infinite" circular well is give by the zeros of the Bessel functions:

$$J_m(kR) = 0, \quad m = 0, \pm 1, \pm 2, \cdots, E = \hbar^2 k^2 / 2m_0$$

Piecewise constant potential

- The Hamiltonian for the circular well problem is

$$H = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m_0} + V(\hat{x}, \hat{y})$$

  where $V(\hat{x}, \hat{y})$ is a "piecewise" constant potential: it is $(-V_0)$ in a circular region of radius R around the origin, and zero elsewhere.

- In the commutative case, the spectrum for the "infinite" circular well is give by the zeros of the Bessel functions:

$$J_m(kR) = 0, \quad m = 0, \pm 1, \pm 2, \cdots, E = \hbar^2 k^2 / 2m_0$$

- In the noncommutative case, the spectrum is very different, and is given by the zeros of the associated Laguerre polynomials:

$$\begin{aligned} L^{m}_{M+1}(\theta k^2/2) &= 0, \quad m \geq 0, \\ L^{|m|}_{M-|m|+1}(\theta k^2/2) &= 0 \quad -M \leq m < 0 \end{aligned}$$

(M is related to the radius: $R^2 = \theta(2M+1)$.

- Spectrum for the noncommutative case is more *sparse* compared to its commutative counterpart.

- In the noncommutative case, the spectrum is very different, and is given by the zeros of the associated Laguerre polynomials:

$$\begin{aligned} L^{m}_{M+1}(\theta k^2/2) &= 0, \quad m \geq 0, \\ L^{|m|}_{M-|m|+1}(\theta k^2/2) &= 0 \quad -M \leq m < 0 \end{aligned}$$

  (M is related to the radius: $R^2 = \theta(2M+1)$.
- Spectrum for the noncommutative case is more *sparse* compared to its commutative counterpart.

Thermodynamics

The most striking effects are brought out by looking at thermodynamics of this system.

- Number of particles in a noncommutative system cannot be made arbitrarily large: there is a “maximal” density!
- Pressure diverges as the maximal density is approached.
- Entropy of the system behaves radically differently: it approaches zero at maximal density.

Thermodynamics

The most striking effects are brought out by looking at thermodynamics of this system.

- Number of particles in a noncommutative system cannot be made arbitrarily large: there is a “maximal” density!
- Pressure diverges as the maximal density is approached.
- Entropy of the system behaves radically differently: it approaches zero at maximal density.

Thermodynamics

The most striking effects are brought out by looking at thermodynamics of this system.

- Number of particles in a noncommutative system cannot be made arbitrarily large: there is a “maximal” density!
- Pressure diverges as the maximal density is approached.
- Entropy of the system behaves radically differently: it approaches zero at maximal density.

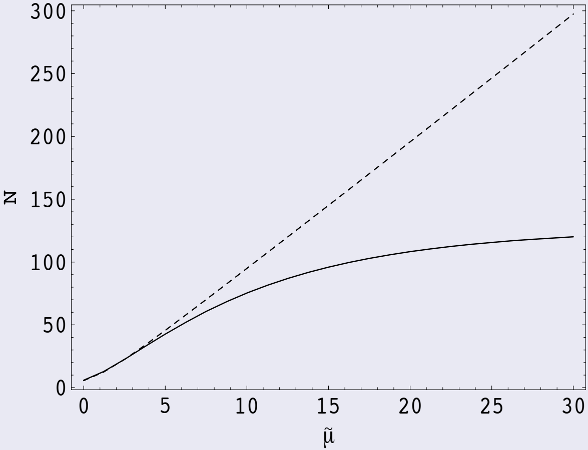
Thermodynamics

The most striking effects are brought out by looking at thermodynamics of this system.

- Number of particles in a noncommutative system cannot be made arbitrarily large: there is a “maximal” density!
- Pressure diverges as the maximal density is approached.
- Entropy of the system behaves radically differently: it approaches zero at maximal density.

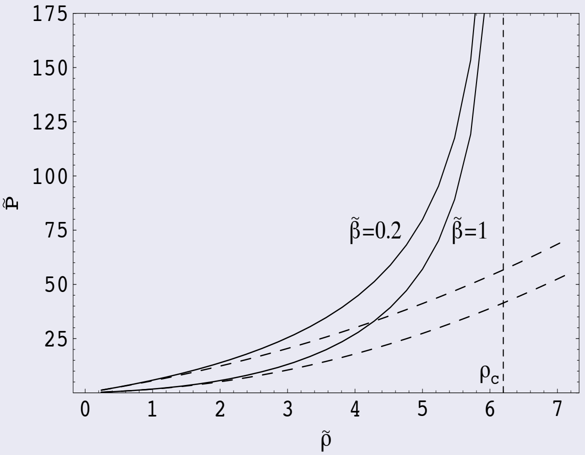
Particle in a noncommutative circular well: thermodynamics

Average number of particles as a function of chemical potential (Dashed line is the commutative case).



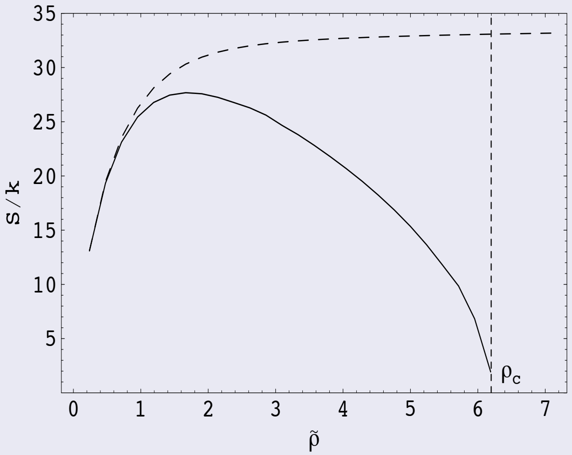
Particle in a noncommutative circular well: thermodynamics

Pressure as a function of density (Dashed line is the commutative case).



Particle in a noncommutative circular well: thermodynamics

Entropy as a function of density (Dashed line is the commutative case).



QFT on commutative spacetime

Before we draw lessons from the quantum mechanical example to carry over to field theory, it is worth recalling some salient aspects of standard Quantum Field Theory (QFT).

- QFT allows us to combine quantum mechanics with the possibility of creating or destroying particles.
- It is also an efficient technology for computing quantities in many-body theory.
- Standard QFT deals with point-like objects.
- When combined with special relativity, standard QFTs also incorporate causality.

QFT on commutative spacetime

Before we draw lessons from the quantum mechanical example to carry over to field theory, it is worth recalling some salient aspects of standard Quantum Field Theory (QFT).

- QFT allows us to combine quantum mechanics with the possibility of creating or destroying particles.
- It is also an efficient technology for computing quantities in many-body theory.
- Standard QFT deals with point-like objects.
- When combined with special relativity, standard QFTs also incorporate causality.

QFT on commutative spacetime

Before we draw lessons from the quantum mechanical example to carry over to field theory, it is worth recalling some salient aspects of standard Quantum Field Theory (QFT).

- QFT allows us to combine quantum mechanics with the possibility of creating or destroying particles.
- It is also an efficient technology for computing quantities in many-body theory.
- Standard QFT deals with point-like objects.
- When combined with special relativity, standard QFTs also incorporate causality.

QFT on commutative spacetime

Before we draw lessons from the quantum mechanical example to carry over to field theory, it is worth recalling some salient aspects of standard Quantum Field Theory (QFT).

- QFT allows us to combine quantum mechanics with the possibility of creating or destroying particles.
- It is also an efficient technology for computing quantities in many-body theory.
- Standard QFT deals with point-like objects.
- When combined with special relativity, standard QFTs also incorporate causality.

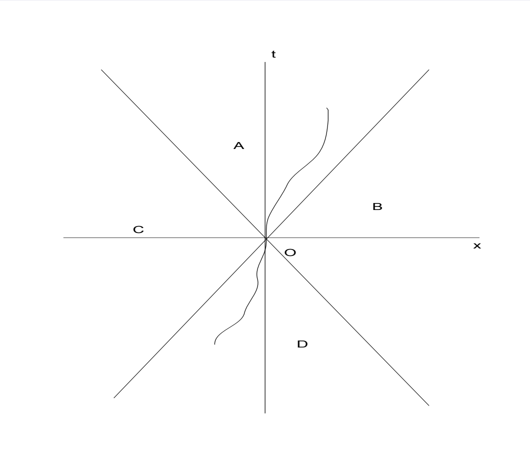
QFT on commutative spacetime

Before we draw lessons from the quantum mechanical example to carry over to field theory, it is worth recalling some salient aspects of standard Quantum Field Theory (QFT).

- QFT allows us to combine quantum mechanics with the possibility of creating or destroying particles.
- It is also an efficient technology for computing quantities in many-body theory.
- Standard QFT deals with point-like objects.
- When combined with special relativity, standard QFTs also incorporate causality.

Light Cone

A is in the future of *O* and *D* is in its past, whereas *B* and *C* are causally unrelated to *O*.



- Mathematically, we say this by requiring that observables ρ at *spacelike* separation commute:

$$[\rho(x), \rho(y)] = 0 \quad \text{if} \quad x \sim y.$$

- This condition is enforced by requiring that the *quantum fields* at points x and y satisfy

$$[\phi(x), \phi(y)]_{\pm} = 0.$$

- Important: these are relativistically invariant statements.

- Mathematically, we say this by requiring that observables ρ at *spacelike* separation commute:

$$[\rho(x), \rho(y)] = 0 \quad \text{if} \quad x \sim y.$$

- This condition is enforced by requiring that the *quantum fields* at points x and y satisfy

$$[\phi(x), \phi(y)]_{\pm} = 0.$$

- Important: these are relativistically invariant statements.

- Mathematically, we say this by requiring that observables ρ at *spacelike* separation commute:

$$[\rho(x), \rho(y)] = 0 \quad \text{if} \quad x \sim y.$$

- This condition is enforced by requiring that the *quantum fields* at points x and y satisfy

$$[\phi(x), \phi(y)]_{\pm} = 0.$$

- Important: these are relativistically invariant statements.

- A deep theorem in QFT: for particles with integer spin, we must choose the minus sign (use the commutator), and for particles with half-integer spin, we must choose the plus sign (use anti-commutator).
- So causality, statistics, and spin are intimately related.
- Relativistic invariance implies that the notions of fermions and bosons are not frame-dependent – e.g. a two-fermion state will be anti-symmetric in all reference frames.

- A deep theorem in QFT: for particles with integer spin, we must choose the minus sign (use the commutator), and for particles with half-integer spin, we must choose the plus sign (use anti-commutator).
- So causality, statistics, and spin are intimately related.
- Relativistic invariance implies that the notions of fermions and bosons are not frame-dependent – e.g. a two-fermion state will be anti-symmetric in all reference frames.

- A deep theorem in QFT: for particles with integer spin, we must choose the minus sign (use the commutator), and for particles with half-integer spin, we must choose the plus sign (use anti-commutator).
- So causality, statistics, and spin are intimately related.
- Relativistic invariance implies that the notions of fermions and bosons are not frame-dependent – e.g. a two-fermion state will be anti-symmetric in all reference frames.

Noncommutative QFT

- Our experience with noncommutative quantum mechanics suggests that particles are not point-like – there is a certain graininess/discreteness.
- Quantum field theories on such a space should somehow retain information of this discreteness.
- We also want relativistic invariance to be compatible with this discreteness – not easy! For example, if we replace $\mathbb{R}^3$ by a discrete lattice, we lose translational and rotational symmetry.

Noncommutative QFT

- Our experience with noncommutative quantum mechanics suggests that particles are not point-like – there is a certain graininess/discreteness.
- Quantum field theories on such a space should somehow retain information of this discreteness.
- We also want relativistic invariance to be compatible with this discreteness – not easy! For example, if we replace $\mathbb{R}^3$ by a discrete lattice, we lose translational and rotational symmetry.

Noncommutative QFT

- Our experience with noncommutative quantum mechanics suggests that particles are not point-like – there is a certain graininess/discreteness.
- Quantum field theories on such a space should somehow retain information of this discreteness.
- We also want relativistic invariance to be compatible with this discreteness – not easy! For example, if we replace $\mathbb{R}^3$ by a discrete lattice, we lose translational and rotational symmetry.

- Fortunately for us, there are mathematical structures (known as twisted Hopf symmetries) that allow us to implement relativistic symmetries on the noncommutative spacetime.
- This procedure of “twisting” can be used to define properties of quantum fields.
- Now particles are not point-like, but have an “extension”.
- The notion of locality and causality become *fuzzy* at very short distances.

- Fortunately for us, there are mathematical structures (known as twisted Hopf symmetries) that allow us to implement relativistic symmetries on the noncommutative spacetime.
- This procedure of "twisting" can be used to define properties of quantum fields.
- Now particles are not point-like, but have an "extension".
- The notion of locality and causality become *fuzzy* at very short distances.

- Fortunately for us, there are mathematical structures (known as twisted Hopf symmetries) that allow us to implement relativistic symmetries on the noncommutative spacetime.
- This procedure of "twisting" can be used to define properties of quantum fields.
- Now particles are not point-like, but have an "extension".
- The notion of locality and causality become *fuzzy* at very short distances.

- Fortunately for us, there are mathematical structures (known as twisted Hopf symmetries) that allow us to implement relativistic symmetries on the noncommutative spacetime.
- This procedure of “twisting” can be used to define properties of quantum fields.
- Now particles are not point-like, but have an “extension”.
- The notion of locality and causality become *fuzzy* at very short distances.

(Naive) Lorentz Transformations

Moyal (or star) product in terms of commutative product

$$
\begin{aligned}
& (f * g)(x) = m_\theta(f \otimes g)(x) = m_0(e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} f \otimes g)(x), \\
= \quad & m_0(\mathcal{F} f \otimes g)(x) = (f \cdot g)(x) + \frac{i}{2}\theta^{\mu\nu}(\partial_\mu f \cdot \partial_\nu g)(x) + \cdots
\end{aligned}
$$

- Under a Lorentz transformation Λ, functions f and g transform as

$$f(x) \to f^\Lambda(x) = f(\Lambda^{-1}x), \quad g(x) \to g^\Lambda(x) = g(\Lambda^{-1}x)$$

$$
\begin{aligned}
(f \cdot g)^\Lambda(x) &= (f^\Lambda \cdot g^\Lambda)(x), \quad \text{BUT} \\
(f * g)^\Lambda(x) &\neq (f^\Lambda * g^\Lambda)(x)!!
\end{aligned}
$$

- Can one do better? Yes, exploiting another underlying algebraic structure.

(Naive) Lorentz Transformations

Moyal (or star) product in terms of commutative product

$$
\begin{aligned}
& (f * g)(x) = m_\theta(f \otimes g)(x) = m_0(e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} f \otimes g)(x), \\
= \quad & m_0(\mathcal{F} f \otimes g)(x) = (f \cdot g)(x) + \frac{i}{2}\theta^{\mu\nu}(\partial_\mu f \cdot \partial_\nu g)(x) + \cdots
\end{aligned}
$$

- Under a Lorentz transformation Λ, functions f and g transform as

$$f(x) \to f^\Lambda(x) = f(\Lambda^{-1}x), \quad g(x) \to g^\Lambda(x) = g(\Lambda^{-1}x)$$

$$
\begin{aligned}
(f \cdot g)^\Lambda(x) \quad &= \quad (f^\Lambda \cdot g^\Lambda)(x), \quad \text{BUT} \\
(f * g)^\Lambda(x) \quad &\neq \quad (f^\Lambda * g^\Lambda)(x)!!
\end{aligned}
$$

- Can one do better? Yes, exploiting another underlying algebraic structure.

(Naive) Lorentz Transformations

Moyal (or star) product in terms of commutative product

$$
\begin{aligned}
& (f * g)(x) = m_\theta(f \otimes g)(x) = m_0(e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} f \otimes g)(x), \\
= \quad & m_0(\mathcal{F} f \otimes g)(x) = (f \cdot g)(x) + \frac{i}{2}\theta^{\mu\nu}(\partial_\mu f \cdot \partial_\nu g)(x) + \cdots
\end{aligned}
$$

- Under a Lorentz transformation Λ, functions f and g transform as

$$
f(x) \rightarrow f^\Lambda(x) = f(\Lambda^{-1}x), \quad g(x) \rightarrow g^\Lambda(x) = g(\Lambda^{-1}x)
$$

$$
\begin{aligned}
(f \cdot g)^\Lambda(x) &= (f^\Lambda \cdot g^\Lambda)(x), \quad \text{BUT} \\
(f * g)^\Lambda(x) &\neq (f^\Lambda * g^\Lambda)(x)!!
\end{aligned}
$$

- Can one do better? Yes, exploiting another underlying algebraic structure.

A Closer Look at the Moyal Algebra

- Left- and right- multiplications are not the same:

$$\hat{x}^L_\mu f = x_\mu * f, \quad \hat{x}^R_\mu f = f * x_\mu.$$

- The left and right actions satisfy:

$$[\hat{x}^L_\mu, \hat{x}^L_\nu] = i\theta_{\mu\nu} = -[\hat{x}^R_\mu, \hat{x}^R_\nu], \quad [\hat{x}^L_\mu, \hat{x}^R_\nu] = 0.$$

- Define (a commuting) $\hat{x}^c_\mu$ in terms $\hat{x}^L_\mu, \hat{x}^R_\mu$ as

$$\hat{x}^c_\mu \equiv \frac{1}{2}\left(\hat{x}^L_\mu + \hat{x}^R_\mu\right), \quad [\hat{x}^c_\mu, \hat{x}^c_\nu] = 0,$$
$$\hat{x}^c_\mu f = \frac{1}{2}(x_\mu * f + f * x_\mu) = x_\mu \cdot f$$

A Closer Look at the Moyal Algebra

- Left- and right- multiplications are not the same:

$$\hat{x}^L_\mu f = x_\mu * f, \quad \hat{x}^R_\mu f = f * x_\mu.$$

- The left and right actions satisfy:

$$[\hat{x}^L_\mu, \hat{x}^L_\nu] = i\theta_{\mu\nu} = -[\hat{x}^R_\mu, \hat{x}^R_\nu], \quad [\hat{x}^L_\mu, \hat{x}^R_\nu] = 0.$$

- Define (a commuting) $\hat{x}^c_\mu$ in terms $\hat{x}^L_\mu, \hat{x}^R_\mu$ as

$$\begin{aligned} \hat{x}^c_\mu &\equiv \frac{1}{2}\left(\hat{x}^L_\mu + \hat{x}^R_\mu\right), \quad [\hat{x}^c_\mu, \hat{x}^c_\nu] = 0, \\ \hat{x}^c_\mu f &= \frac{1}{2}(x_\mu * f + f * x_\mu) = x_\mu \cdot f \end{aligned}$$

A Closer Look at the Moyal Algebra

- Left- and right- multiplications are not the same:

$$\hat{x}^L_\mu f = x_\mu * f, \quad \hat{x}^R_\mu f = f * x_\mu.$$

- The left and right actions satisfy:

$$[\hat{x}^L_\mu, \hat{x}^L_\nu] = i\theta_{\mu\nu} = -[\hat{x}^R_\mu, \hat{x}^R_\nu], \quad [\hat{x}^L_\mu, \hat{x}^R_\nu] = 0.$$

- Define (a commuting) $\hat{x}^c_\mu$ in terms $\hat{x}^L_\mu, \hat{x}^R_\mu$ as

$$\begin{aligned} \hat{x}^c_\mu &\equiv \frac{1}{2}\left(\hat{x}^L_\mu + \hat{x}^R_\mu\right), \quad [\hat{x}^c_\mu, \hat{x}^c_\nu] = 0, \\ \hat{x}^c_\mu f &= \frac{1}{2}\left(x_\mu * f + f * x_\mu\right) = x_\mu \cdot f \end{aligned}$$

A Second Look at Lorentz Transformations

- Under Λ, $f(x) \to f^{\Lambda}(x) = f(\Lambda^{-1}x)$. This is an operation on a single function, and does not require the star (or any) product.
- Under an infinitesimal transformation $\Lambda \simeq \mathbf{1} + i\epsilon^{\mu\nu}M_{\mu\nu}$,
$$f^{\Lambda}(x) \simeq f(x) - i\epsilon^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)f(x).$$
Notice that in the above, there is no star!
- So $M_{\mu\nu} = \hat{x}^c_\mu \hat{p}_\nu - \hat{x}^c_\nu \hat{p}_\mu \qquad (\hat{p}_\mu = -i\partial_\mu)$
- Actually, this is how an arbitrary vector field also acts on noncommutative functions: $\hat{v}f = [v(\hat{x}^c_\mu)\partial_\mu f](x)$.
- These generate infinitesimal diffeos, now making it possible to discuss gravity theories.

A Second Look at Lorentz Transformations

- Under Λ, $f(x) \to f^{\Lambda}(x) = f(\Lambda^{-1}x)$. This is an operation on a single function, and does not require the star (or any) product.
- Under an infinitesimal transformation $\Lambda \simeq \mathbf{1} + i\epsilon^{\mu\nu}M_{\mu\nu}$,
$$f^{\Lambda}(x) \simeq f(x) - i\epsilon^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)f(x).$$
Notice that in the above, there is no star!
- So $M_{\mu\nu} = \hat{x}^c_\mu\hat{p}_\nu - \hat{x}^c_\nu\hat{p}_\mu \qquad (\hat{p}_\mu = -i\partial_\mu)$
- Actually, this is how an arbitrary vector field also acts on noncommutative functions: $\hat{v}f = [v(\hat{x}^c_\mu)\partial_\mu f](x)$.
- These generate infinitesimal diffeos, now making it possible to discuss gravity theories.

A Second Look at Lorentz Transformations

- Under Λ, $f(x) \to f^{\Lambda}(x) = f(\Lambda^{-1}x)$. This is an operation on a single function, and does not require the star (or any) product.
- Under an infinitesimal transformation $\Lambda \simeq \mathbf{1} + i\epsilon^{\mu\nu}M_{\mu\nu}$,
$$f^{\Lambda}(x) \simeq f(x) - i\epsilon^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)f(x).$$
Notice that in the above, there is no star!
- So $M_{\mu\nu} = \hat{x}^c_\mu \hat{p}_\nu - \hat{x}^c_\nu \hat{p}_\mu \qquad (\hat{p}_\mu = -i\partial_\mu)$
- Actually, this is how an arbitrary vector field also acts on noncommutative functions: $\hat{v}f = [v(\hat{x}^c_\mu)\partial_\mu f](x)$.
- These generate infinitesimal diffeos, now making it possible to discuss gravity theories.

A Second Look at Lorentz Transformations

- Under Λ, $f(x) \to f^{\Lambda}(x) = f(\Lambda^{-1}x)$. This is an operation on a single function, and does not require the star (or any) product.
- Under an infinitesimal transformation $\Lambda \simeq \mathbf{1} + i\epsilon^{\mu\nu}M_{\mu\nu}$,
$$f^{\Lambda}(x) \simeq f(x) - i\epsilon^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)f(x).$$
Notice that in the above, there is no star!
- So $M_{\mu\nu} = \hat{x}^c_\mu \hat{p}_\nu - \hat{x}^c_\nu \hat{p}_\mu \qquad (\hat{p}_\mu = -i\partial_\mu)$
- Actually, this is how an arbitrary vector field also acts on noncommutative functions: $\hat{v}f = [v(\hat{x}^c_\mu)\partial_\mu f](x)$.
- These generate infinitesimal diffeos, now making it possible to discuss gravity theories.

A Second Look at Lorentz Transformations

- Under Λ, $f(x) \to f^{\Lambda}(x) = f(\Lambda^{-1}x)$. This is an operation on a single function, and does not require the star (or any) product.
- Under an infinitesimal transformation $\Lambda \simeq \mathbf{1} + i\epsilon^{\mu\nu}M_{\mu\nu}$,
$$f^{\Lambda}(x) \simeq f(x) - i\epsilon^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)f(x).$$
Notice that in the above, there is no star!
- So $M_{\mu\nu} = \hat{x}^c_\mu\hat{p}_\nu - \hat{x}^c_\nu\hat{p}_\mu \qquad (\hat{p}_\mu = -i\partial_\mu)$
- Actually, this is how an arbitrary vector field also acts on noncommutative functions: $\hat{v}f = [v(\hat{x}^c_\mu)\partial_\mu f](x)$.
- These generate infinitesimal diffeos, now making it possible to discuss gravity theories.

Modified Leibnitz rule

- Although the $M_{\mu\nu}$ correctly generate the Lorentz algebra, their action on the star product of two functions is different:

$$
\begin{aligned}
& M_{\mu\nu}(\alpha * \beta) = (M_{\mu\nu}\alpha) * \beta + \alpha * (M_{\mu\nu}\beta) \\
- \quad & \frac{1}{2}\left[((\hat{p} \cdot \theta)_{\mu}\alpha) * (\hat{p}_{\nu}\beta) - (\hat{p}_{\nu}\alpha) * ((\hat{p} \cdot \theta)_{\mu}\beta) - \mu \leftrightarrow \nu\right], \\
& (\hat{p} \cdot \theta)_{\rho} := \hat{p}_{\lambda}\theta^{\lambda}_{\rho}.
\end{aligned}
$$

- This mysterious modification has its origins in Hopf algebra theory (Drinfel'd).

Modified Leibnitz rule

- Although the $M_{\mu\nu}$ correctly generate the Lorentz algebra, their action on the star product of two functions is different:

$$
\begin{aligned}
& M_{\mu\nu}(\alpha * \beta) = (M_{\mu\nu}\alpha) * \beta + \alpha * (M_{\mu\nu}\beta) \\
- \quad & \frac{1}{2}\left[((\hat{p}\cdot\theta)_\mu\alpha) * (\hat{p}_\nu\beta) - (\hat{p}_\nu\alpha) * ((\hat{p}\cdot\theta)_\mu\beta) - \mu \leftrightarrow \nu\right], \\
& (\hat{p}\cdot\theta)_\rho := \hat{p}_\lambda\theta^\lambda_\rho .
\end{aligned}
$$

- This mysterious modification has its origins in Hopf algebra theory (Drinfel'd).

Group/Algebra Action on Vector Spaces

Suppose a group G acts on a vector space V as

Group Action

$$v \to \rho(g)v, \quad v \in V, \text{ and } \rho \text{ a representation of } G.$$

- On a tensor product $V \otimes W$, the group acts as

$$g : (v \otimes w) \to (\rho_1(g)v) \otimes (\rho_2(g)w).$$

- So we need a map (a coproduct) Δ which "splits" g so that it can act on tensor products.

Group/Algebra Action on Vector Spaces

Suppose a group G acts on a vector space V as

Group Action

$$v \to \rho(g)v, \quad v \in V, \text{ and } \rho \text{ a representation of } G.$$

- On a tensor product $V \otimes W$, the group acts as

$$g : (v \otimes w) \to (\rho_1(g)v) \otimes (\rho_2(g)w).$$

- So we need a map (a coproduct) Δ which "splits" g so that it can act on tensor products.

Group/Algebra Action on Vector Spaces

Suppose a group G acts on a vector space V as

Group Action

$$v \to \rho(g)v, \quad v \in V, \text{ and } \rho \text{ a representation of } G.$$

- On a tensor product $V \otimes W$, the group acts as

$$g : (v \otimes w) \to (\rho_1(g)v) \otimes (\rho_2(g)w).$$

- So we need a map (a coproduct) Δ which "splits" g so that it can act on tensor products.

Now an escalation

- Suppose V is also an algebra (we can multiply two vectors to get another vector).
- Then our coproduct better be compatible with multiplication in V!
- First multiplying v and w, and then acting on the product by g, must be the same as first transforming v and w separately by g and then multiplying them.
- If such a coproduct Δ exists, we say that G is an automorphism of the algebra.

Now an escalation

- Suppose V is also an algebra (we can multiply two vectors to get another vector).
- Then our coproduct better be compatible with multiplication in V!
- First multiplying v and w, and then acting on the product by g, must be the same as first transforming v and w separately by g and then multiplying them.
- If such a coproduct Δ exists, we say that G is an automorphism of the algebra.

Now an escalation

- Suppose V is also an algebra (we can multiply two vectors to get another vector).
- Then our coproduct better be compatible with multiplication in V!
- First multiplying v and w, and then acting on the product by g, must be the same as first transforming v and w separately by g and then multiplying them.
- If such a coproduct Δ exists, we say that G is an automorphism of the algebra.

Now an escalation

- Suppose V is also an algebra (we can multiply two vectors to get another vector).
- Then our coproduct better be compatible with multiplication in V!
- First multiplying v and w, and then acting on the product by g, must be the same as first transforming v and w separately by g and then multiplying them.
- If such a coproduct Δ exists, we say that G is an automorphism of the algebra.

More On Coproduct

- The usual coproduct $\Delta_0(\Lambda) = \Lambda \times \Lambda$ is compatible ordinary multiplication, but not with Moyal multiplication.
- But a twisted coproduct Δ_θ defined as
$$\Delta_\theta(\Lambda) = \mathcal{F}^{-1}\Delta_0(\Lambda)\mathcal{F}$$
is compatible with Moyal product!
- Indeed, $m_\theta[\Delta_\theta(\Lambda) f \otimes g] = \rho(\Lambda) m_\theta(f \otimes g)$.
- For infinitesimal Lorentz transformations, the twisted coproduct reproduces our earlier result:

$$\begin{aligned}\Delta_\theta(M_{\mu\nu}) &= M_{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M_{\mu\nu} \\ &- \frac{1}{2}\left[(\hat{p}\cdot\theta)_\mu \otimes \hat{p}_\nu - \hat{p}_\nu \otimes (\hat{p}\cdot\theta)_\mu - (\mu \leftrightarrow \nu)\right]\end{aligned}$$

More On Coproduct

- The usual coproduct $\Delta_0(\Lambda) = \Lambda \times \Lambda$ is compatible ordinary multiplication, but not with Moyal multiplication.
- But a twisted coproduct Δ_θ defined as
$$\Delta_\theta(\Lambda) = \mathcal{F}^{-1}\Delta_0(\Lambda)\mathcal{F}$$
is compatible with Moyal product!
- Indeed, $m_\theta[\Delta_\theta(\Lambda) f \otimes g] = \rho(\Lambda) m_\theta(f \otimes g)$.
- For infinitesimal Lorentz transformations, the twisted coproduct reproduces our earlier result:

$$\begin{aligned}\Delta_\theta(M_{\mu\nu}) &= M_{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M_{\mu\nu} \\ &- \frac{1}{2}\left[(\hat{p}\cdot\theta)_\mu \otimes \hat{p}_\nu - \hat{p}_\nu \otimes (\hat{p}\cdot\theta)_\mu - (\mu \leftrightarrow \nu)\right]\end{aligned}$$

More On Coproduct

- The usual coproduct $\Delta_0(\Lambda) = \Lambda \times \Lambda$ is compatible ordinary multiplication, but not with Moyal multiplication.
- But a twisted coproduct Δ_θ defined as
$$\Delta_\theta(\Lambda) = \mathcal{F}^{-1}\Delta_0(\Lambda)\mathcal{F}$$
is compatible with Moyal product!
- Indeed, $m_\theta[\Delta_\theta(\Lambda) f \otimes g] = \rho(\Lambda) m_\theta(f \otimes g)$.
- For infinitesimal Lorentz transformations, the twisted coproduct reproduces our earlier result:

$$\begin{aligned}\Delta_\theta(M_{\mu\nu}) &= M_{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M_{\mu\nu} \\ &- \frac{1}{2}\left[(\hat{p}\cdot\theta)_\mu \otimes \hat{p}_\nu - \hat{p}_\nu \otimes (\hat{p}\cdot\theta)_\mu - (\mu \leftrightarrow \nu)\right]\end{aligned}$$

More On Coproduct

- The usual coproduct $\Delta_0(\Lambda) = \Lambda \times \Lambda$ is compatible ordinary multiplication, but not with Moyal multiplication.
- But a twisted coproduct Δ_θ defined as
$$\Delta_\theta(\Lambda) = \mathcal{F}^{-1}\Delta_0(\Lambda)\mathcal{F}$$
is compatible with Moyal product!
- Indeed, $m_\theta[\Delta_\theta(\Lambda) f \otimes g] = \rho(\Lambda) m_\theta(f \otimes g)$.
- For infinitesimal Lorentz transformations, the twisted coproduct reproduces our earlier result:

$$\begin{aligned} \Delta_\theta(M_{\mu\nu}) &= M_{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M_{\mu\nu} \\ &- \frac{1}{2}\left[(\hat{p}\cdot\theta)_\mu \otimes \hat{p}_\nu - \hat{p}_\nu \otimes (\hat{p}\cdot\theta)_\mu - (\mu \leftrightarrow \nu)\right] \end{aligned}$$

Implications for Quantum Statistics

- In usual quantum mechanics, the wavefunction of two identical particles is the (anti-)symmetrized tensor product of single particle wavefunctions:

$$\phi \otimes_{S,A} \chi \equiv \frac{1}{2}(\phi \otimes \chi \pm \chi \otimes \phi) = \left(\frac{1 \pm \tau_0}{2}\right)(\phi \otimes \chi)$$

- The flip operator τ_0 is superselected: all observables (including $M_{\mu\nu}$) commute with it.
- However, $\tau_0 \, \Delta_\theta(\Lambda) \neq \Delta_\theta(\Lambda)\tau_0$, so usual (anti-)symmetrization is incompatible with the new coproduct.

Implications for Quantum Statistics

- In usual quantum mechanics, the wavefunction of two identical particles is the (anti-)symmetrized tensor product of single particle wavefunctions:

$$\phi \otimes_{S,A} \chi \equiv \frac{1}{2}(\phi \otimes \chi \pm \chi \otimes \phi) = \left(\frac{1 \pm \tau_0}{2}\right)(\phi \otimes \chi)$$

- The flip operator τ_0 is superselected: all observables (including $M_{\mu\nu}$) commute with it.
- However, $\tau_0 \, \Delta_\theta(\Lambda) \neq \Delta_\theta(\Lambda)\tau_0$, so usual (anti-)symmetrization is incompatible with the new coproduct.

Implications for Quantum Statistics

- In usual quantum mechanics, the wavefunction of two identical particles is the (anti-)symmetrized tensor product of single particle wavefunctions:

$$\phi \otimes_{S,A} \chi \equiv \frac{1}{2}(\phi \otimes \chi \pm \chi \otimes \phi) = \left(\frac{1 \pm \tau_0}{2}\right)(\phi \otimes \chi)$$

- The flip operator τ_0 is superselected: all observables (including $M_{\mu\nu}$) commute with it.
- However, $\tau_0 \, \Delta_\theta(\Lambda) \neq \Delta_\theta(\Lambda)\tau_0$, so usual (anti-)symmetrization is incompatible with the new coproduct.

Implications for Quantum Statistics

- But a twisted flip operator $\tau_\theta \equiv \mathcal{F}^{-1} \tau_0 \mathcal{F}$ does: this changes the notion of fermions/bosons.
- The states constructed according to

$$\phi \otimes_{S_\theta} \chi \equiv \left(\frac{1 + \tau_\theta}{2} \right) (\phi \otimes \chi), \quad \phi \otimes_{A_\theta} \chi \equiv \left(\frac{1 - \tau_\theta}{2} \right) (\phi \otimes \chi)$$

form the physical two-particle Hilbert spaces of (generalized) bosons and fermions and obey twisted statistics.

Implications for Quantum Statistics

- But a twisted flip operator $\tau_\theta \equiv \mathcal{F}^{-1}\tau_0\mathcal{F}$ does: this changes the notion of fermions/bosons.
- The states constructed according to

$$\phi \otimes_{S_\theta} \chi \equiv \left(\frac{1+\tau_\theta}{2}\right)(\phi \otimes \chi), \quad \phi \otimes_{A_\theta} \chi \equiv \left(\frac{1-\tau_\theta}{2}\right)(\phi \otimes \chi)$$

  form the physical two-particle Hilbert spaces of (generalized) bosons and fermions and obey twisted statistics.

Twisted Quantum Fields

- Suppose $\Phi(x)$ is a second-quantized field, and $a_p^\dagger$ the creation operator with momentum p. As usual we require that

$$
\begin{aligned}
\langle 0|\Phi^{(-)}(x)a_p^\dagger|0\rangle &= e_p(x), \\
\langle 0|\Phi^{(-)}(x_1)\Phi^{(-)}(x_2)a_q^\dagger a_p^\dagger|0\rangle &= (\mathbf{1} \pm \tau_\theta)\,(e_p \otimes e_q)(x_1, x_2) \\
&\equiv (e_p \otimes_{S_\theta, A_\theta} e_q)(x_1, x_2)
\end{aligned}
$$

- This gives us the twisted commutation relations:

$$
\begin{aligned}
a_p^\dagger\, a_q^\dagger &= \pm e^{i\theta^{\mu\nu} p_\mu q_\nu}\, a_q^\dagger\, a_p^\dagger \\
a_p a_q &= \pm e^{i\theta^{\mu\nu} p_\mu q_\nu}\, a_q a_p
\end{aligned}
$$

Twisted Quantum Fields

- Suppose $\Phi(x)$ is a second-quantized field, and $a_p^\dagger$ the creation operator with momentum p. As usual we require that

$$
\begin{aligned}
\langle 0|\Phi^{(-)}(x)a_p^\dagger|0\rangle &= e_p(x), \\
\langle 0|\Phi^{(-)}(x_1)\Phi^{(-)}(x_2)a_q^\dagger a_p^\dagger|0\rangle &= (\mathbf{1} \pm \tau_\theta)\,(e_p \otimes e_q)(x_1, x_2) \\
&\equiv (e_p \otimes_{S_\theta, A_\theta} e_q)(x_1, x_2)
\end{aligned}
$$

- This gives us the twisted commutation relations:

$$
\begin{aligned}
a_p^\dagger\, a_q^\dagger &= \pm e^{i\theta^{\mu\nu} p_\mu q_\nu}\, a_q^\dagger\, a_p^\dagger \\
a_p a_q &= \pm e^{i\theta^{\mu\nu} p_\mu q_\nu}\, a_q a_p
\end{aligned}
$$

Twisted Quantum Fields

- Interestingly, we can realize the twisted operators $a_p, a_p^\dagger$ in terms on usual Fock space operators $c_p, c_p^\dagger$:

$$c_p c_q - c_q c_p = 0, \quad c_p c_q^\dagger - c_q^\dagger c_p = 2p_0 \delta(\vec{p} - \vec{q}).$$

$$a_p = c_p e^{\frac{i}{2} p_\mu \theta^{\mu\nu} \mathcal{P}_\nu} \equiv c_p e^{\frac{i}{2} p \wedge \mathcal{P}},$$

where $\mathcal{P}_\nu = \int d\mu(k) k_\nu c_k^\dagger c_k$ is the usual Fock space momentum operator.

- This in turn allows us to write

$$\Phi(x) = \Phi^c(x) e^{\frac{1}{2} \overleftarrow{\partial} \wedge \mathcal{P}}$$

Twisted Quantum Fields

- Interestingly, we can realize the twisted operators $a_p, a_p^\dagger$ in terms on usual Fock space operators $c_p, c_p^\dagger$:

$$c_p c_q - c_q c_p = 0, \quad c_p c_q^\dagger - c_q^\dagger c_p = 2p_0 \delta(\vec{p} - \vec{q}).$$

$$a_p = c_p e^{\frac{i}{2} p_\mu \theta^{\mu\nu} \mathcal{P}_\nu} \equiv c_p e^{\frac{i}{2} p \wedge \mathcal{P}},$$

where $\mathcal{P}_\nu = \int d\mu(k) k_\nu c_k^\dagger c_k$ is the usual Fock space momentum operator.

- This in turn allows us to write

$$\Phi(x) = \Phi^c(x) e^{\frac{1}{2} \overleftarrow{\partial} \wedge \mathcal{P}}$$

Simple Implications of Twisted Statistics

- The notion of identical particles can still be defined, but now there is a scale dependence – for example, a two-fermion wavefunction is anti-symmetric at low energies, but picks us a symmetric piece at high energies.
- Consider a two-fermion state

$$|\alpha, \beta\rangle = \int d\mu)p_1)d\mu(p_2)\alpha(p_1)\beta(p_2)a^\dagger(p_1)a^\dagger(p_2)|0\rangle$$

  Notice that $|\alpha, \alpha\rangle$ does not vanish!
- This is an example of a "Pauli-forbidden" state.
- An experimental signature would be a transition between a "Pauli-allowed" and a "Pauli-forbidden" state.

Simple Implications of Twisted Statistics

- The notion of identical particles can still be defined, but now there is a scale dependence – for example, a two-fermion wavefunction is anti-symmetric at low energies, but picks us a symmetric piece at high energies.
- Consider a two-fermion state

$$|\alpha, \beta\rangle = \int d\mu)p_1)d\mu(p_2)\alpha(p_1)\beta(p_2)a^\dagger(p_1)a^\dagger(p_2)|0\rangle$$

Notice that $|\alpha, \alpha\rangle$ does not vanish!

- This is an example of a "Pauli-forbidden" state.
- An experimental signature would be a transition between a "Pauli-allowed" and a "Pauli-forbidden" state.

Simple Implications of Twisted Statistics

- The notion of identical particles can still be defined, but now there is a scale dependence – for example, a two-fermion wavefunction is anti-symmetric at low energies, but picks us a symmetric piece at high energies.
- Consider a two-fermion state

$$|\alpha, \beta\rangle = \int d\mu)p_1)d\mu(p_2)\alpha(p_1)\beta(p_2)a^\dagger(p_1)a^\dagger(p_2)|0\rangle$$

  Notice that $|\alpha, \alpha\rangle$ does not vanish!
- This is an example of a "Pauli-forbidden" state.
- An experimental signature would be a transition between a "Pauli-allowed" and a "Pauli-forbidden" state.

Simple Implications of Twisted Statistics

- The notion of identical particles can still be defined, but now there is a scale dependence – for example, a two-fermion wavefunction is anti-symmetric at low energies, but picks us a symmetric piece at high energies.
- Consider a two-fermion state

$$|\alpha, \beta\rangle = \int d\mu)p_1)d\mu(p_2)\alpha(p_1)\beta(p_2)a^\dagger(p_1)a^\dagger(p_2)|0\rangle$$

  Notice that $|\alpha, \alpha\rangle$ does not vanish!
- This is an example of a "Pauli-forbidden" state.
- An experimental signature would be a transition between a "Pauli-allowed" and a "Pauli-forbidden" state.

Simple Implications of Twisted Statistics

- In Borexino and SuperKamiokande experiments, one can look for forbidden transitions from O^{16} to $\tilde{O}^{16}$ where the tilde nuclei have an extra nucleon in the filled $1S_{1/2}$ level.
- Their lifetime is greater than 10^{27} years.
- Experiments done on forbidden transitions to filled K-shells of crystals (Maryland group) – branching ratios less than 10^{-25} for such transitions.
- These give $\sqrt{\theta} < 10^{-26}$ m.

Simple Implications of Twisted Statistics

- In Borexino and SuperKamiokande experiments, one can look for forbidden transitions from O^{16} to $\tilde{O}^{16}$ where the tilde nuclei have an extra nucleon in the filled $1S_{1/2}$ level.
- Their lifetime is greater than 10^{27} years.
- Experiments done on forbidden transitions to filled K-shells of crystals (Maryland group) – branching ratios less than 10^{-25} for such transitions.
- These give $\sqrt{\theta} < 10^{-26}$ m.

Simple Implications of Twisted Statistics

- In Borexino and SuperKamiokande experiments, one can look for forbidden transitions from O^{16} to $\tilde{O}^{16}$ where the tilde nuclei have an extra nucleon in the filled $1S_{1/2}$ level.
- Their lifetime is greater than 10^{27} years.
- Experiments done on forbidden transitions to filled K-shells of crystals (Maryland group) – branching ratios less than 10^{-25} for such transitions.
- These give $\sqrt{\theta} < 10^{-26}$ m.

Simple Implications of Twisted Statistics

- In Borexino and SuperKamiokande experiments, one can look for forbidden transitions from O^{16} to $\tilde{O}^{16}$ where the tilde nuclei have an extra nucleon in the filled $1S_{1/2}$ level.
- Their lifetime is greater than 10^{27} years.
- Experiments done on forbidden transitions to filled K-shells of crystals (Maryland group) – branching ratios less than 10^{-25} for such transitions.
- These give $\sqrt{\theta} < 10^{-26}$ m.

Gauge transformations

- Gauge fields A_λ transform as one-forms under diffeos generated by vector fields. They could be functions of $\hat{x}^c$ or $\hat{x}^L$.
- If $A_\lambda = A_\lambda(\hat{x}^c)$, then we can write gauge theories for arbitrary gauge groups. These theories are identical to the corresponding commutative ones.
- If $A_\lambda = A_\lambda(\hat{x}^L)$, then we can only construct $U(N)$ gauge theories.

Gauge transformations

- Gauge fields A_λ transform as one-forms under diffeos generated by vector fields. They could be functions of $\hat{x}^c$ or $\hat{x}^L$.
- If $A_\lambda = A_\lambda(\hat{x}^c)$, then we can write gauge theories for arbitrary gauge groups. These theories are identical to the corresponding commutative ones.
- If $A_\lambda = A_\lambda(\hat{x}^L)$, then we can only construct $U(N)$ gauge theories.

Gauge transformations

- Gauge fields A_λ transform as one-forms under diffeos generated by vector fields. They could be functions of $\hat{x}^c$ or $\hat{x}^L$.
- If $A_\lambda = A_\lambda(\hat{x}^c)$, then we can write gauge theories for arbitrary gauge groups. These theories are identical to the corresponding commutative ones.
- If $A_\lambda = A_\lambda(\hat{x}^L)$, then we can only construct $U(N)$ gauge theories.

Gauge Covariant Derivatives

- Under a gauge transformation $g(\hat{x}^c)$, a charged matter field $\Phi(x)$ transforms as $\Phi(x) \to g(x)\Phi(x)$.
- The quantum covariant derivative D_μ must respect this *module* property of the gauge group:

$$D_\mu(g\Phi) = gD_\mu\Phi + (\partial_\mu g)\Phi$$

- D_μ must also respect (twisted) statistics, and Poincaré covariance.
- The only one which does this is

$$D_\mu\Phi = (D^c_\mu\Phi^c)e^{\frac{1}{2}\overleftarrow{\partial}_\mu\theta^{\mu\nu}\mathcal{P}_\nu}$$

Gauge Covariant Derivatives

- Under a gauge transformation $g(\hat{x}^c)$, a charged matter field $\Phi(x)$ transforms as $\Phi(x) \to g(x)\Phi(x)$.
- The quantum covariant derivative D_μ must respect this *module* property of the gauge group:

$$D_\mu(g\Phi) = gD_\mu\Phi + (\partial_\mu g)\Phi$$

- D_μ must also respect (twisted) statistics, and Poincaré covariance.
- The only one which does this is

$$D_\mu\Phi = (D_\mu^c\Phi^c)e^{\frac{1}{2}\overleftarrow{\partial}_\mu\theta^{\mu\nu}\mathcal{P}_\nu}$$

Gauge Covariant Derivatives

- Under a gauge transformation $g(\hat{x}^c)$, a charged matter field $\Phi(x)$ transforms as $\Phi(x) \to g(x)\Phi(x)$.
- The quantum covariant derivative D_μ must respect this *module* property of the gauge group:

$$D_\mu(g\Phi) = gD_\mu\Phi + (\partial_\mu g)\Phi$$

- D_μ must also respect (twisted) statistics, and Poincaré covariance.
- The only one which does this is

$$D_\mu\Phi = (D^c_\mu\Phi^c)e^{\frac{1}{2}\overleftarrow{\partial}_\mu\theta^{\mu\nu}\mathcal{P}_\nu}$$

Gauge Covariant Derivatives

- Under a gauge transformation $g(\hat{x}^c)$, a charged matter field $\Phi(x)$ transforms as $\Phi(x) \to g(x)\Phi(x)$.
- The quantum covariant derivative D_μ must respect this *module* property of the gauge group:

$$D_\mu(g\Phi) = gD_\mu\Phi + (\partial_\mu g)\Phi$$

- D_μ must also respect (twisted) statistics, and Poincaré covariance.
- The only one which does this is

$$D_\mu\Phi = (D_\mu^c\Phi^c)e^{\frac{1}{2}\overleftarrow{\partial}_\mu\theta^{\mu\nu}\mathcal{P}_\nu}$$

Gauge Field Strength

- Field strength is the commutator of two covariant derivatives:

$$[D_\mu, D_\nu]\Phi = ([D^c_\mu, D^c_\nu]\Phi^c)e^{\frac{1}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu}\mathcal{P}_\nu} = (F^c_{\mu\nu}\Phi^c)e^{\frac{1}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu}\mathcal{P}_\nu}$$

- $F^c_{\mu\nu}$ transforms correctly under gauge transformations, and can be used to construct the Hamiltonian for quantum gauge theory.
- Pure gauge theory is identical to its commutative counterpart, but not when matter is included.

Gauge Field Strength

- Field strength is the commutator of two covariant derivatives:

$$[D_\mu, D_\nu]\Phi = ([D^c_\mu, D^c_\nu]\Phi^c)e^{\frac{1}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu}\mathcal{P}_\nu} = (F^c_{\mu\nu}\Phi^c)e^{\frac{1}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu}\mathcal{P}_\nu}$$

- $F^c_{\mu\nu}$ transforms correctly under gauge transformations, and can be used to construct the Hamiltonian for quantum gauge theory.
- Pure gauge theory is identical to its commutative counterpart, but not when matter is included.

Gauge Field Strength

- Field strength is the commutator of two covariant derivatives:

$$[D_\mu, D_\nu]\Phi = ([D^c_\mu, D^c_\nu]\Phi^c)e^{\frac{1}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu}\mathcal{P}_\nu} = (F^c_{\mu\nu}\Phi^c)e^{\frac{1}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu}\mathcal{P}_\nu}$$

- $F^c_{\mu\nu}$ transforms correctly under gauge transformations, and can be used to construct the Hamiltonian for quantum gauge theory.
- Pure gauge theory is identical to its commutative counterpart, but not when matter is included.

Matter-Gauge Interactions

- The interaction Hamiltonian is of the form

$$\begin{aligned} H^I_\theta &= \int d^3x[\mathcal{H}^{MG}_\theta + \mathcal{H}^G_\theta], \\ \mathcal{H}^{MG}_\theta &= \mathcal{H}^{MG}_0 e^{\frac{1}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu} \mathcal{P}_\nu}, \\ \mathcal{H}^G_\theta &= \mathcal{H}^G_0 \end{aligned}$$

  $\mathcal{H}^{MG}$ has all matter-matter and matter gauge couplings, $\mathcal{H}^G$ has only gauge field terms.

- For non-Abelian theories, cross-terms between $\mathcal{H}^{MG}$ and $\mathcal{H}^G$ lead to Lorentz-violating effects (QCD or Standard Model).

Matter-Gauge Interactions

- The interaction Hamiltonian is of the form

$$
\begin{aligned}
H^I_\theta &= \int d^3x [\mathcal{H}^{MG}_\theta + \mathcal{H}^G_\theta], \\
\mathcal{H}^{MG}_\theta &= \mathcal{H}^{MG}_0 e^{\frac{1}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \mathcal{P}_\nu}, \\
\mathcal{H}^G_\theta &= \mathcal{H}^G_0
\end{aligned}
$$

  $\mathcal{H}^{MG}$ has all matter-matter and matter gauge couplings, $\mathcal{H}^G$ has only gauge field terms.
- For non-Abelian theories, cross-terms between $\mathcal{H}^{MG}$ and $\mathcal{H}^G$ lead to Lorentz-violating effects (QCD or Standard Model).

Signatures of Noncommutativity

- Any high-energy phenomenon involving identical particles is expected to carry signatures of noncommutativity, through the deformation of the spin-statistics connection.
- New physics at high densities – potential implications for neutron star physics, Chandrasekhar limit, and early cosmology.
- Non-abelian gauge theories break relativistic invariance (and also CPT theorem) at the quantum level. These can give unique signatures in particle scattering processes.
- Physics that involves a sharp separation of spacetime into regions gets affected – black hole physics.

Signatures of Noncommutativity

- Any high-energy phenomenon involving identical particles is expected to carry signatures of noncommutativity, through the deformation of the spin-statistics connection.
- New physics at high densities – potential implications for neutron star physics, Chandrasekhar limit, and early cosmology.
- Non-abelian gauge theories break relativistic invariance (and also CPT theorem) at the quantum level. These can give unique signatures in particle scattering processes.
- Physics that involves a sharp separation of spacetime into regions gets affected – black hole physics.

Signatures of Noncommutativity

- Any high-energy phenomenon involving identical particles is expected to carry signatures of noncommutativity, through the deformation of the spin-statistics connection.
- New physics at high densities – potential implications for neutron star physics, Chandrasekhar limit, and early cosmology.
- Non-abelian gauge theories break relativistic invariance (and also CPT theorem) at the quantum level. These can give unique signatures in particle scattering processes.
- Physics that involves a sharp separation of spacetime into regions gets affected – black hole physics.

Signatures of Noncommutativity

- Any high-energy phenomenon involving identical particles is expected to carry signatures of noncommutativity, through the deformation of the spin-statistics connection.
- New physics at high densities – potential implications for neutron star physics, Chandrasekhar limit, and early cosmology.
- Non-abelian gauge theories break relativistic invariance (and also CPT theorem) at the quantum level. These can give unique signatures in particle scattering processes.
- Physics that involves a sharp separation of spacetime into regions gets affected – black hole physics.

Cosmic Rays

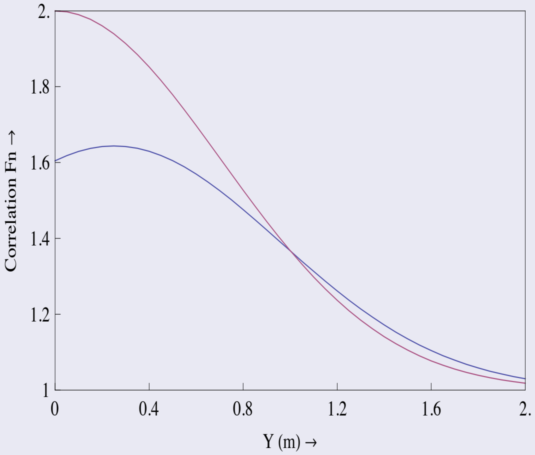
- Cosmic rays are typically extreme high energy protons (of energies as high as $10^{19} eV$) that which collide with the earth's atmosphere to produce a shower of secondary particles.
- By studying two-particle distrubution function, it is possible to obtain a bound on θ.

Cosmic Rays

- Cosmic rays are typically extreme high energy protons (of energies as high as $10^{19} eV$) that which collide with the earth's atmosphere to produce a shower of secondary particles.
- By studying two-particle distrubution function, it is possible to obtain a bound on θ.

Two-particle distribution function

The commutative distribution (red) and the noncommutative distribution (blue), with $\sqrt{\theta} \sim 10^{-13} m$



QED from Spontaneously Broken $SU(2) \times U(1)$

- The gauge group for the Standard Model is non-Abelian, and will show similar effects.
- In particular, signatures of Lorentz (or spin-statistics) violation can be seen in QED.
- A simple test is to look at the scattering at identical fermions: in usual quantum theory, this amplitude vanishes at 90^0 scattering.

QED from Spontaneously Broken $SU(2) \times U(1)$

- The gauge group for the Standard Model is non-Abelian, and will show similar effects.
- In particular, signatures of Lorentz (or spin-statistics) violation can be seen in QED.
- A simple test is to look at the scattering at identical fermions: in usual quantum theory, this amplitude vanishes at 90^0 scattering.

QED from Spontaneously Broken $SU(2) \times U(1)$

- The gauge group for the Standard Model is non-Abelian, and will show similar effects.
- In particular, signatures of Lorentz (or spin-statistics) violation can be seen in QED.
- A simple test is to look at the scattering at identical fermions: in usual quantum theory, this amplitude vanishes at 90^0 scattering.

Möller scattering in QED

- The interaction Hamiltonian is $H_I = \frac{e}{2}\int d^3x[\bar{\psi}(x) * \left(\not{A}(\hat{x}^c)\psi(x)\right) + h.c.]$
- We can calculate the scattering amplitude $\mathcal{T}_\theta$ in the centre-of-momentum frame, with the spins of the electrons aligned. It depends on scattering angle Θ_M, dimensionless c.m energy $x = E/m$, and $t = m^2\theta_{ij}\epsilon^{ijk}(\hat{p}_F \times \hat{p}_I)^k$.

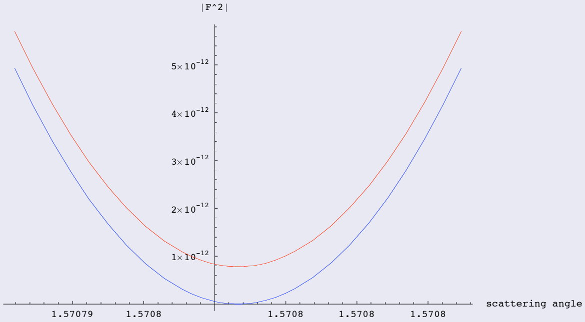
Möller scattering in QED

- The interaction Hamiltonian is
$H_I = \frac{e}{2}\int d^3x[\bar{\psi}(x) * \left(\not{A}(\hat{x}^c)\psi(x)\right) + h.c.]$
- We can calculate the scattering amplitude $\mathcal{T}_\theta$ in the centre-of-momentum frame, with the spins of the electrons aligned. It depends on scattering angle Θ_M, dimensionless c.m energy $x = E/m$, and $t = m^2\theta_{ij}\epsilon^{ijk}(\hat{p}_F \times \hat{p}_I)^k$.

Möller scattering in QED

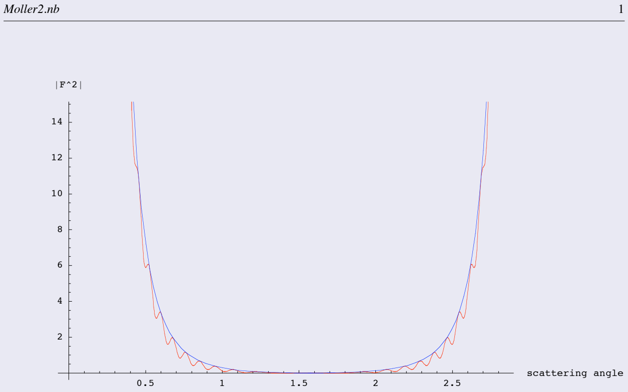
Normalized scattering cross-section for $t = 10^{-5}$ and $x = 100$

Moller1.nb 1



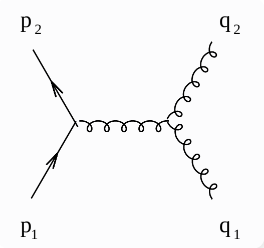
Möller scattering in QED

Normalized scattering cross-section for $t = 10^{-2}$ and $x = 100$



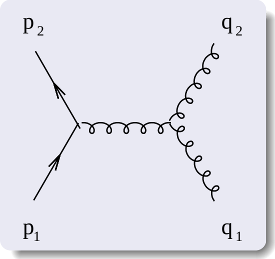
Non-Abelian Gauge Theories

- In non-Abelian gauge theories, there are even more (conceptually) dramatic effects: these theories lose relativistic invariance at the quantum level.
- Processes like $qg \to qg$ (quark-gluon scattering) violate Lorentz invariance (the propagator is "frame-dependent").



Non-Abelian Gauge Theories

- In non-Abelian gauge theories, there are even more (conceptually) dramatic effects: these theories lose relativistic invariance at the quantum level.
- Processes like $qg \to qg$ (quark-gluon scattering) violate Lorentz invariance (the propagator is “frame-dependent”).



Summary

- Noncommutative spacetime can be thought of as the bridge between low-energy quantum field theory, and the (eventual) theory of quantum gravity.
- By taking advantage of new algebraic structures (twists) from Hopf algebra theory, it is indeed possible to discuss Lorentz-invariant QFT's on noncommutative space.
- Twisting deforms statistics of identical particles, with possible signatures for Pauli principle violation at high energies.
- Noncommutative non-Abelian gauge theories show even more dramatic effects, with the S-matrix for some processes violating Lorentz invariance.
- Signatures for spin-statistics violation may be extracted from QED.

Summary

- Noncommutative spacetime can be thought of as the bridge between low-energy quantum field theory, and the (eventual) theory of quantum gravity.
- By taking advantage of new algebraic structures (twists) from Hopf algebra theory, it is indeed possible to discuss Lorentz-invariant QFT's on noncommutative space.
- Twisting deforms statistics of identical particles, with possible signatures for Pauli principle violation at high energies.
- Noncommutative non-Abelian gauge theories show even more dramatic effects, with the S-matrix for some processes violating Lorentz invariance.
- Signatures for spin-statistics violation may be extracted from QED.

Summary

- Noncommutative spacetime can be thought of as the bridge between low-energy quantum field theory, and the (eventual) theory of quantum gravity.
- By taking advantage of new algebraic structures (twists) from Hopf algebra theory, it is indeed possible to discuss Lorentz-invariant QFT's on noncommutative space.
- Twisting deforms statistics of identical particles, with possible signatures for Pauli principle violation at high energies.
- Noncommutative non-Abelian gauge theories show even more dramatic effects, with the S-matrix for some processes violating Lorentz invariance.
- Signatures for spin-statistics violation may be extracted from QED.

Summary

- Noncommutative spacetime can be thought of as the bridge between low-energy quantum field theory, and the (eventual) theory of quantum gravity.
- By taking advantage of new algebraic structures (twists) from Hopf algebra theory, it is indeed possible to discuss Lorentz-invariant QFT's on noncommutative space.
- Twisting deforms statistics of identical particles, with possible signatures for Pauli principle violation at high energies.
- Noncommutative non-Abelian gauge theories show even more dramatic effects, with the S-matrix for some processes violating Lorentz invariance.
- Signatures for spin-statistics violation may be extracted from QED.

Summary

- Noncommutative spacetime can be thought of as the bridge between low-energy quantum field theory, and the (eventual) theory of quantum gravity.
- By taking advantage of new algebraic structures (twists) from Hopf algebra theory, it is indeed possible to discuss Lorentz-invariant QFT's on noncommutative space.
- Twisting deforms statistics of identical particles, with possible signatures for Pauli principle violation at high energies.
- Noncommutative non-Abelian gauge theories show even more dramatic effects, with the S-matrix for some processes violating Lorentz invariance.
- Signatures for spin-statistics violation may be extracted from QED.

Future Directions

- Outlook
  - Spontaneous Symmetry breaking can also be discussed in this framework. This will give us the noncommutative Standard Model, and phenomenological signatures.
  - Noncommutativity makes the lightcone structure "fuzzy", leading to leakage of signals across lightlike horizons.
  - Twisted fermi statistics change the equation of state for a "free" fermi gas – implications for early cosmology.
  - Julius Wess and his collaborators have extensively developed classical tensor analysis using this as a starting point, including a noncommutative version of the classical Einstein action for gravity. The solutions of this Einstein theory are still largely unexplored.

Future Directions

- Outlook
  - Spontaneous Symmetry breaking can also be discussed in this framework. This will give us the noncommutative Standard Model, and phenomenological signatures.
  - Noncommutativity makes the lightcone structure "fuzzy", leading to leakage of signals across lightlike horizons.
  - Twisted fermi statistics change the equation of state for a "free" fermi gas – implications for early cosmology.
  - Julius Wess and his collaborators have extensively developed classical tensor analysis using this as a starting point, including a noncommutative version of the classical Einstein action for gravity. The solutions of this Einstein theory are still largely unexplored.

Future Directions

- Outlook
  - Spontaneous Symmetry breaking can also be discussed in this framework. This will give us the noncommutative Standard Model, and phenomenological signatures.
  - Noncommutativity makes the lightcone structure “fuzzy”, leading to leakage of signals across lightlike horizons.
  - Twisted fermi statistics change the equation of state for a “free” fermi gas – implications for early cosmology.
  - Julius Wess and his collaborators have extensively developed classical tensor analysis using this as a starting point, including a noncommutative version of the classical Einstein action for gravity. The solutions of this Einstein theory are still largely unexplored.

Future Directions

- Outlook
  - Spontaneous Symmetry breaking can also be discussed in this framework. This will give us the noncommutative Standard Model, and phenomenological signatures.
  - Noncommutativity makes the lightcone structure “fuzzy”, leading to leakage of signals across lightlike horizons.
  - Twisted fermi statistics change the equation of state for a “free” fermi gas – implications for early cosmology.
  - Julius Wess and his collaborators have extensively developed classical tensor analysis using this as a starting point, including a noncommutative version of the classical Einstein action for gravity. The solutions of this Einstein theory are still largely unexplored.

Future Directions

- Outlook
  - Spontaneous Symmetry breaking can also be discussed in this framework. This will give us the noncommutative Standard Model, and phenomenological signatures.
  - Noncommutativity makes the lightcone structure “fuzzy”, leading to leakage of signals across lightlike horizons.
  - Twisted fermi statistics change the equation of state for a “free” fermi gas – implications for early cosmology.
  - Julius Wess and his collaborators have extensively developed classical tensor analysis using this as a starting point, including a noncommutative version of the classical Einstein action for gravity. The solutions of this Einstein theory are still largely unexplored.

Collaborators I

- A. P. Balachandran, Sasha Pinzul, Babar Qureshi
- Biswajit Chakraborty, Frederik Scholtz, Jan Goevarts
- T. R. Govindarajan, Giampiero Mangano
- Fedele Lizzi, Patrizia Vitale
- At CHEP: Nitin Chandra, Rahul Srivastava, Nirmalendu Acharyya (graduate students), and Prasad Bose (post-doc).

References

- arXiv:hep-th/0508002
- arXix:hep-th/0601056, arXiv:hep-th/0608138, arXiv:hep-th/0608179
- arXiv:0707.3858 [hep-th], arXiv:0708.0069 [hep-th], arXiv:0708.1379 [hep-th], arXiv:0709.3357 [hep-th]
- arXiv:0811.2050 [quant-ph]
- arXiv:0901.1712 [hep-th]