Periods of Calabi-Yau Manifolds in Physics and Number Theory

TIFR 15 March 2011

Periods of Calabi-Yau Manifolds in Physics and Number Theory

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Aims

- To explain why the periods of CY manifolds are important to string theorists.
- These periods are also important to number theorists because they encode important arithmetic information about the manifold.
- I want to speculate about the role of `quantum corrections' and mirror symmetry for the zeta-function.

CY manifolds arise in the "compactification" of string theory from D=10 down to D=4. A desirable consequence of string theory is spacetime supersymmetry and the question arises as to whether this process preserves supersymmetry.

$$\begin{aligned} \delta b &= f\,\epsilon \\ \delta f &= b\,\epsilon + \nabla\epsilon \end{aligned}$$

Very quickly we come to the condition

$$\nabla\epsilon = 0\ .$$

Which we take as a condition on the manifold

$$[\nabla, \nabla]\epsilon = 0\ .$$

Given ϵ we construct

$$J_m{}^n = -\mathrm{i}\epsilon^\dagger\gamma_m{}^n\epsilon \qquad\qquad \Omega_{mnr} = \epsilon^T\gamma_{mnr}\,\epsilon$$

$$
h^{pq} \;=\; \begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{11} & & 0 & \\
1 & & h^{21} & & h^{21} & & 1 \\
 & 0 & & h^{11} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

An example is the quintic threefold:

$$
\mathcal{M} \;:\; P(x,\psi) \;=\; \sum_{i=1}^{5} x_i^5 - 5\psi\, x_1 x_2 x_3 x_4 x_5 \;.
$$

This has $h^{11} = 1$ and $h^{21} = 101$.

$$h^{pq} \;=\; \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{11} & & 0 & \\ 1 & & h^{21} & & h^{21} & & 1 \\ & 0 & & h^{11} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

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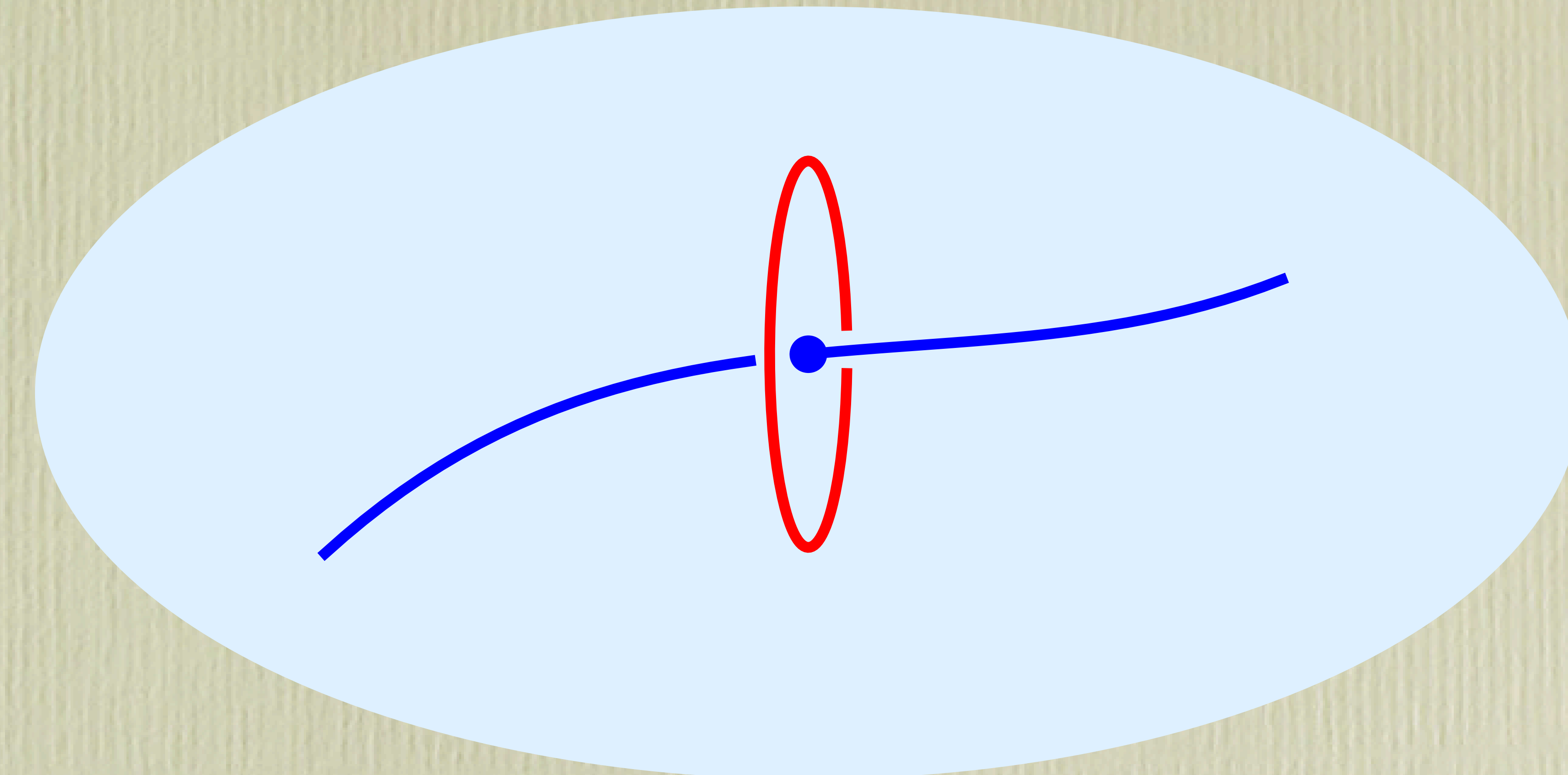
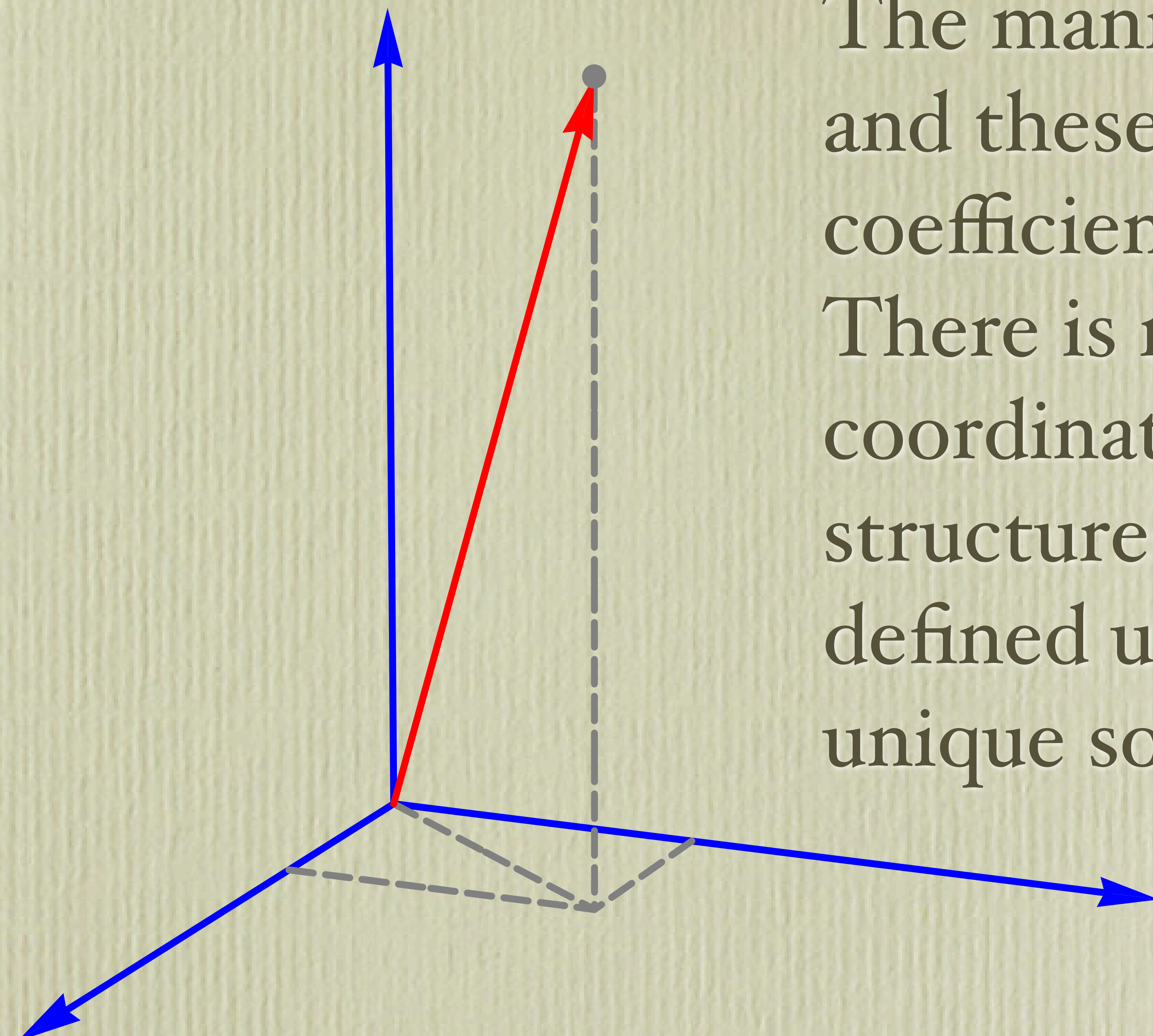


The existence of the holomorphic form is a very useful fact since the form can be constructed explicitly

$$\Omega = \frac{1}{2\pi\mathrm{i}} \oint \frac{\epsilon_{abcde}\, x^a dx^b dx^c dx^d dx^e}{P}$$

$$= \epsilon_{abcde} \frac{x^a dx^b dx^c dx^d}{\frac{\partial P}{\partial x^e}}$$



The manifold depends on parameters and these may be thought of as the coefficients in the defining polynomials. There is more canonical way to give coordinates on the space of complex structures. The holomorphic form is defined up to scale but is otherwise unique so defines a line in $H^3(\mathcal{M}, \mathbb{Z})$.

The coordinates of this line are

$$\varpi_\Gamma = \int_\Gamma \Omega$$

taken over a basis of cycles $\Gamma \in H_3(\mathcal{M}, \mathbb{Z})$ and these are the periods.

We may think of the following polynomial as defining a one-parameter subfamily of the 101 parameter space of quintics

$$\mathcal{M} \;:\; P(x,\psi) \;=\; \sum_{i=1}^{5} x_i^5 - 5\psi\, x_1 x_2 x_3 x_4 x_5 \;.$$

In this case there is a simple relation between this family and the one-parameter mirror family

$$\mathcal{W} \;=\; \widehat{\mathcal{M}/\Gamma}$$

$$\Gamma \;:\; (x_1, x_2, x_3, x_4, x_5) \;\mapsto\; (\zeta^{n_1} x_1, \zeta^{n_2} x_2, \zeta^{n_3} x_3, \zeta^{n_4} x_4, \zeta^{n_5} x_5)$$

where $\zeta^5 = 1$ and $\sum_i n_i \equiv 0 \bmod 5$

Periods for the mirror quintic

Consider Ω for the mirror quintic family, which has just one complex structure parameter so

$$b^3 = 1 + h^{21} + h^{21} + 1 = 4$$

The quantities

$$\Omega,\ \Omega',\ \Omega'',\ \Omega''',\ \Omega''''$$

are all 3-forms, and there are five of them, so there is a linear relation

$$\mathcal{L}\Omega = 0$$

$$
\begin{aligned}
\varpi &= -\frac{5\psi}{(2\pi\mathrm{i})^3}\int_{\gamma_2\times\gamma_3\times\gamma_4}\frac{x^1dx^2dx^3dx^4}{\frac{\partial P}{\partial x^5}} \\
&= -\frac{5\psi}{(2\pi\mathrm{i})^4}\int_{\gamma_2\times\gamma_3\times\gamma_4\times\gamma_5}\frac{x^1dx^2dx^3dx^4dx^5}{P} \\
&= -\frac{5\psi}{(2\pi\mathrm{i})^5}\int_{\gamma_1\times\gamma_2\times\gamma_3\times\gamma_4\times\gamma_5}\frac{dx^1dx^2dx^3dx^4dx^5}{P} \\
&= \frac{1}{(2\pi\mathrm{i})^5}\int_\Gamma\frac{dx^1dx^2dx^3dx^4dx^5}{x^1x^2x^3x^4x^5\left(1-\frac{\sum_i(x^i)^5}{5\psi\,x^1x^2x^3x^4x^5}\right)} \\
&= \frac{1}{(2\pi\mathrm{i})^5}\int_\Gamma\frac{dx^1dx^2dx^3dx^4dx^5}{x^1x^2x^3x^4x^5}\sum_{m=0}^{\infty}\left(\frac{(x^1)^5+\ldots+(x^5)^5}{5\psi\,x^1x^2x^3x^4x^5}\right)^m \\
&= \sum_{n=0}^{\infty}\frac{(5n)!}{(n!)^5}\frac{1}{(5\psi)^{5n}}
\end{aligned}
$$

$$\frac{1}{2\pi i}\int \frac{dx^1}{x^1}$$

$$\begin{aligned}
\varpi &= -\frac{5\psi}{(2\pi\mathrm{i})^3}\int_{\gamma_2\times\gamma_3\times\gamma_4} \frac{x^1 dx^2 dx^3 dx^4}{\frac{\partial P}{\partial x^5}} \\
&= -\frac{5\psi}{(2\pi\mathrm{i})^4}\int_{\gamma_2\times\gamma_3\times\gamma_4\times\gamma_5} \frac{x^1 dx^2 dx^3 dx^4 dx^5}{P} \\
&= -\frac{5\psi}{(2\pi\mathrm{i})^5}\int_{\gamma_1\times\gamma_2\times\gamma_3\times\gamma_4\times\gamma_5} \frac{dx^1 dx^2 dx^3 dx^4 dx^5}{P} \\
&= \frac{1}{(2\pi\mathrm{i})^5}\int_{\Gamma} \frac{dx^1 dx^2 dx^3 dx^4 dx^5}{x^1x^2x^3x^4x^5\left(1-\frac{\sum_i (x^i)^5}{5\psi\, x^1x^2x^3x^4x^5}\right)} \\
&= \frac{1}{(2\pi\mathrm{i})^5}\int_{\Gamma} \frac{dx^1 dx^2 dx^3 dx^4 dx^5}{x^1x^2x^3x^4x^5}\sum_{m=0}^{\infty}\left(\frac{(x^1)^5+\ldots+(x^5)^5}{5\psi\, x^1x^2x^3x^4x^5}\right)^m \\
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\end{aligned}
$$

$$\frac{1}{2\pi i}\int\frac{dx^1}{x^1}$$

$$
\begin{aligned}
&= -\frac{5\psi}{(2\pi \mathrm{i})^4}\int_{\gamma_2\times\gamma_3\times\gamma_4\times\gamma_5}\frac{x^1\,dx^2dx^3dx^4dx^5}{P}\\
&= -\frac{5\psi}{(2\pi \mathrm{i})^5}\int_{\gamma_1\times\gamma_2\times\gamma_3\times\gamma_4\times\gamma_5}\frac{dx^1dx^2dx^3dx^4dx^5}{P}\\
&= \frac{1}{(2\pi \mathrm{i})^5}\int_{\Gamma}\frac{dx^1dx^2dx^3dx^4dx^5}{x^1x^2x^3x^4x^5\left(1-\frac{\sum_i (x^i)^5}{5\psi\, x^1x^2x^3x^4x^5}\right)}
\end{aligned}
$$

$$5n$$

$$
\begin{aligned}
&= \frac{1}{(2\pi \mathrm{i})^5}\int_{\Gamma}\frac{dx^1dx^2dx^3dx^4dx^5}{x^1x^2x^3x^4x^5}\sum_{m=0}^{\infty}\left(\frac{(x^1)^5+\ldots+(x^5)^5}{5\psi\, x^1x^2x^3x^4x^5}\right)^m\\
&= \sum_{n=0}^{\infty}\frac{(5n)!}{(n!)^5}\,\frac{1}{(5\psi)^{5n}}
\end{aligned}
$$

Tuesday, 22 March 2011

We expect the period to satisfy a fourth order differential equation, and indeed it does

$$\mathcal{L}\varpi = 0 \;; \quad \varpi(\lambda) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5}\, \lambda^n \;, \quad \lambda = \frac{1}{(5\psi)^5}$$

where

$$\mathcal{L} = \vartheta^4 - 5\lambda \prod_{i=1}^{4} (5\vartheta + i) \;; \quad \vartheta = \lambda \frac{d}{d\lambda} \;.$$

The point $\lambda = 0$ is a regular singular point with all four indices equal to zero. Thus the solutions near the origin are asymptotic to

$$1,\ \log \lambda,\ \log^2 \lambda,\ \log^3 \lambda$$

More precisely the solutions are of the form

$$
\begin{aligned}
\varpi_0(\lambda) &= f_0(\lambda) \\
\varpi_1(\lambda) &= f_0(\lambda)\,\log\lambda + f_1(\lambda) \\
\varpi_2(\lambda) &= f_0(\lambda)\,\log^2\lambda + 2f_1(\lambda)\,\log\lambda + f_2(\lambda) \\
\varpi_3(\lambda) &= f_0(\lambda)\,\log^3\lambda + 3f_1(\lambda)\,\log^2\lambda + 3f_2(\lambda)\,\log\lambda + f_3(\lambda)
\end{aligned}
$$

where the $f_j(\lambda)$ are power series. These series will enter into the Yukawa couplings and into the calculation of the numbers of $\mathbb{F}_p$-rational points of $\mathcal{M}$. Recall that these solutions may be found by the method of Fröbenius. That is by seeking solutions

$$\varpi(\lambda,\varepsilon) \;=\; \sum_{m=0}^{\infty} a_m(\varepsilon)\,\lambda^{m+\varepsilon} \text{ to the eqn. } \mathcal{L}\,\varpi(\lambda,\varepsilon) \;=\; \varepsilon^4\lambda^\varepsilon\ .$$

Yukawa couplings and integral series

This is the expression for the Yukawa coupling

$$y_{ttt} \;=\; 5\left(\frac{2\pi\mathrm{i}}{5}\right)^3 \frac{\psi^2}{\varpi_0(\psi)^2(1-\psi^5)}\left(\frac{d\psi}{dt}\right)^3 \;=\; 5+\sum_{k=0}^{\infty}\frac{n_k k^3 q^k}{1-q^k}\;,$$

where, in these expressions

$$t=\frac{1}{2\pi\mathrm{i}}\,\frac{\varpi_1(\lambda)}{\varpi_0(\lambda)} \quad \text{and} \quad q=\exp(2\pi\mathrm{i}t)$$

$$t = \frac{1}{2\pi\mathrm{i}}\,\frac{\varpi_1(\lambda)}{\varpi_0(\lambda)} \qquad\qquad q = \exp(2\pi\mathrm{i}t)$$

These relations give the mirror map. Note that integers arise here also

$$\begin{aligned}\lambda \quad = \quad & q + 154\,q^2 + 179139\,q^3 + 313195944\,q^4 \\ & +657313805125\,q^5 + 1531113959577750\,q^6 \\ & +3815672803541261385\,q^7 \\ & +9970002717955633142112\,q^8 + \dots\ .\end{aligned}$$

Arithmetic of the Quintic

Now ask what is, for a physicist, a very strange question: For the quintic

$$\sum_{i=1}^{5} x_i^5 - 5\psi\, x_1 x_2 x_3 x_4 x_5 = 0$$

How many solutions to this equation are there with integer x^k and how does this number vary with ψ?

Since the x^k are coordinates in a projective space we are free to multiply by a common scale, so there is no difference between a solution in integers and a solution in rational numbers. This formulation is better because $\mathbb{Q}$ is a field but is still hard.

Easier: How many solutions are there over a finite field?

Field Theory for Physicists

A field $\mathbb{F}$ is a set on which $+$ and $\times$ are defined and have the usual associative and distributive properties.

$\mathbb{F}$ is an abelian group with respect to and $+$
$\mathbb{F}^* = \mathbb{F} \setminus \{0\}$ is an abelian group with respect to $\times$.

Finite fields are uniquely classified by the number of elements, which is p^N for some prime p and integer N.

$\mathbb{F}_7$							
x	0	1	2	3	4	5	6
x^{-1}	$*$	1	4	5	2	3	6

An old result, going back to Fermat, is $a^p = a$, write this as

$$a(a^{p-1} - 1) = 0 \quad \text{it follows that} \quad a^{p-1} = \begin{cases} 1, & \text{if } a \neq 0 \\ 0, & \text{if } a = 0\ . \end{cases}$$

There is another elementary fact that is very useful

$$\sum_{a \in \mathbb{F}_p} a^n \ = \ \sum_{a \in \mathbb{F}_p} (ba)^n \ = \ b^n \sum_{a \in \mathbb{F}_p} a^n\ .$$

It follows that, for $a \neq 0$,

$$\sum_{a \in \mathbb{F}_p} a^n \ = \ \begin{cases} 0, & \text{if } p-1 \text{ does not divide } n \\ -1, & \text{if } p-1 \text{ divides } n\ . \end{cases}$$

A Zero'th Order Result

Take now $x \in \mathbb{F}_p^5$ and $5\psi \in \mathbb{F}_p\,,\ p \neq 5$ and let

$$\nu_\lambda \;=\; \#\{x \mid P(x,\psi)=0\}, \quad \lambda = \frac{1}{(5\psi)^5}$$

This number can be computed mod p with relative ease.

$$\nu_\lambda \;\equiv\; \sum_{x\in\mathbb{F}_p^5} \left(1 - P(x,\psi)^{p-1}\right)$$

$$\nu_\lambda \;\equiv\; {}^{[p/5]}\varpi_0(\lambda) \;=\; \sum_{m=0}^{[p/5]} \frac{(5m)!}{(m!)^5}\,\lambda^m\ .$$

p-Adic Numbers

ν_λ is a definite number so we may seek to compute it exactly.

$$\nu_\lambda \;=\; \nu_\lambda^{(0)} + \nu_\lambda^{(1)}\, p + \nu_\lambda^{(2)}\, p^2 + \nu_\lambda^{(3)}\, p^3 + \nu_\lambda^{(4)}\, p^4$$

with $0 \le \nu_\lambda^{(j)} \le p-1$ and evaluate $\mathrm{mod}\, p^2$, $\mathrm{mod}\, p^3$ and so on. This leads naturally to p-adic analysis. Given $r \in \mathbb{Q}$ we write

$$r \;=\; \frac{m}{n} \;=\; \frac{m_0}{n_0}\, p^\alpha$$

The p-adic norm of r is defined by

$$\|r\|_p \;=\; p^{-\alpha}\,, \quad \|0\|_p \;=\; 0$$

Counting the Points Exactly

The result is most simply stated for the case $5 \not| \, p-1$

$$\nu_\lambda \;=\; {}^p f_0(\Lambda) + \left(\frac{p}{1-p}\right) {}^p f_1'(\Lambda) + \frac{1}{2!}\left(\frac{p}{1-p}\right)^2 {}^p f_2''(\Lambda)$$

$$+\frac{1}{3!}\left(\frac{p}{1-p}\right)^3 {}^p f_3'''(\Lambda) + \frac{1}{4!}\left(\frac{p}{1-p}\right)^4 {}^p f_4''''(\Lambda) + \mathcal{O}(p^5)$$

In this expression

$\Lambda = \mathrm{Teich}(\lambda) = \lim\limits_{n\to\infty} \lambda^{p^n}$ and ${}^p f_0(\Lambda) \;=\; \displaystyle\sum_{m=0}^{p-1} \frac{(5m)!}{(m!)^5}\,\Lambda^m$

We can also perform the sum in this expression for the number of points

$$\nu_\lambda \;=\; \sum_{m=0}^{p-1} \beta_m \, \Lambda^m$$

with coefficients

$$\beta_m \;=\; \lim_{n\to\infty} \frac{a_{m(1+p+p^2+\ldots+p^{n+1})}}{a_{m(1+p+p^2+\ldots+p^n)}} \;=\; (-1)^m \, G_{5m} \, G^5_{-m}$$

Mirror Symmetry and the Zeta Function

We work now over $\mathbb{F}_{p^r}$ and denote by $N_r(\psi)$ the number of projective solutions to the equation $P(x,\psi) = 0$.

$$\zeta(T,\psi) \;=\; \exp\left(\sum_{r=1}^{\infty} \frac{N_r(\psi)T^r}{r}\right)$$

$$= \frac{\text{Numerator of deg. } 2h^{21}+2 \text{ dep. on the cpx. structure of } \mathcal{M}}{\text{Denominator of deg. } 2h^{11}+2}\;.$$

Explicitly for the quintic we have

$$\zeta_{\mathcal{M}}(T,\psi) \;=\; \frac{R_0(T,\psi)\, R_{\mathcal{A}}(p^\rho T^\rho,\psi)^{\frac{20}{\rho}}\; R_{\mathcal{B}}(p^\rho T^\rho,\psi)^{\frac{30}{\rho}}}{(1-T)(1-pT)(1-p^2T)(1-p^3T)}$$

$$\zeta_{\mathcal{W}}(T,\psi) \;=\; \frac{R_0(T,\psi)}{(1-T)(1-pT)^{101}(1-p^2T)^{101}(1-p^3T)}$$

where $\rho = 1,\ 2,\ 4$ is the smallest integer such that $5|(p^\rho - 1)$

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The 5-adic Limit

The desired relations are true in the 5-adic limit. For all p and ψ

$$R_0(T,\psi) = (1-T)(1-pT)(1-p^2T)(1-p^3T) + \mathcal{O}(5^2)$$

$$R_{\mathcal{A}}(T,\psi)^{20} R_{\mathcal{B}}(T,\psi)^{30} = (1-pT)^{100}(1-p^2T)^{100} + \mathcal{O}(5^2)$$

Compare this with the quantum corrections to the classical Yukawa coupling

$$\frac{y_{ttt}}{y_{ttt}^{(0)}} \;=\; 1 + \frac{1}{5}\sum_{k=0}^{\infty} \frac{n_k k^3 q^k}{1-q^k} \;=\; 1 + \mathcal{O}(5^2)$$

since Lian and Yau have shown that $5^3 | n_k k^3$ for each k.