ELECTROMAGNETIC SHOWERS

Paolo Lipari

Corsika school
Ooty 19th december 2010

Lecture 3

AVERAGE LONGITUDINAL EVOLUTION

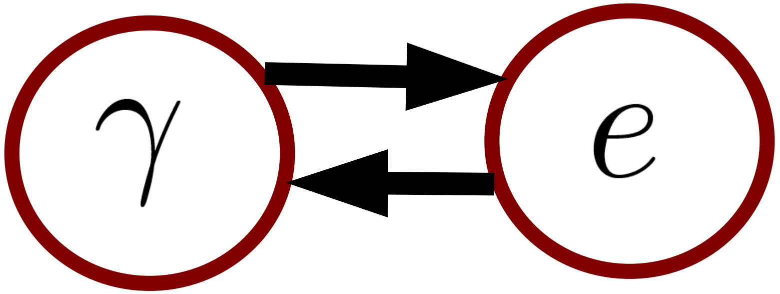
For a

PURELY ELECTROMAGNETIC SHOWER

$$n_e(E, t)$$

$$n_\gamma(E, t)$$

Two functions of energy and depth



In what follows we shall call "*approximation A*" the approximation in which collision processes and Compton effect are neglected, and the asymptotic formulae are used to describe radiation processes and pair production.

A

We shall call "*approximation B*" the approximation in which the Compton effect is neglected, the collision loss is described as a constant energy dissipation and the asymptotic formulae for radiation processes and pair production are used.

B

$$\frac{\partial n_e(E,t)}{\partial t} = -\int_0^1 dv \ \varphi(v) \left[n_e(E,t) - \frac{1}{1-v}\, n_e\left(\frac{E}{1-v}, t\right)\right]$$

$$+2 \int_0^1 \frac{du}{u}\, \psi(u)\, n_\gamma\left(\frac{E}{u}, t\right)$$

$$\frac{\partial n_\gamma(E,t)}{\partial t} = \int_0^1 \frac{dv}{v}\, \varphi(v)\, n_e\left(\frac{E}{v}, t\right) - \sigma_0\, n_\gamma(E,t)$$

Approximation A

$$\frac{\partial n_e(E,t)}{\partial t} = -\int_0^1 dv \;\; \varphi(v) \left[n_e(E,t) - \frac{1}{1-v}\, n_e\left(\frac{E}{1-v}, t\right)\right]$$

$$+2 \int_0^1 \frac{du}{u}\, \psi(u)\, n_\gamma\left(\frac{E}{u}, t\right)$$

$$+\,\varepsilon\, \frac{\partial n_e(E,t)}{\partial E}$$

$$\frac{\partial n_\gamma(E,t)}{\partial t} = \int_0^1 \frac{dv}{v}\, \varphi(v)\, n_e\left(\frac{E}{v}, t\right) - \sigma_0\, n_\gamma(E,t)$$

Approximation B

“Elementary Solutions”
Approximation A:

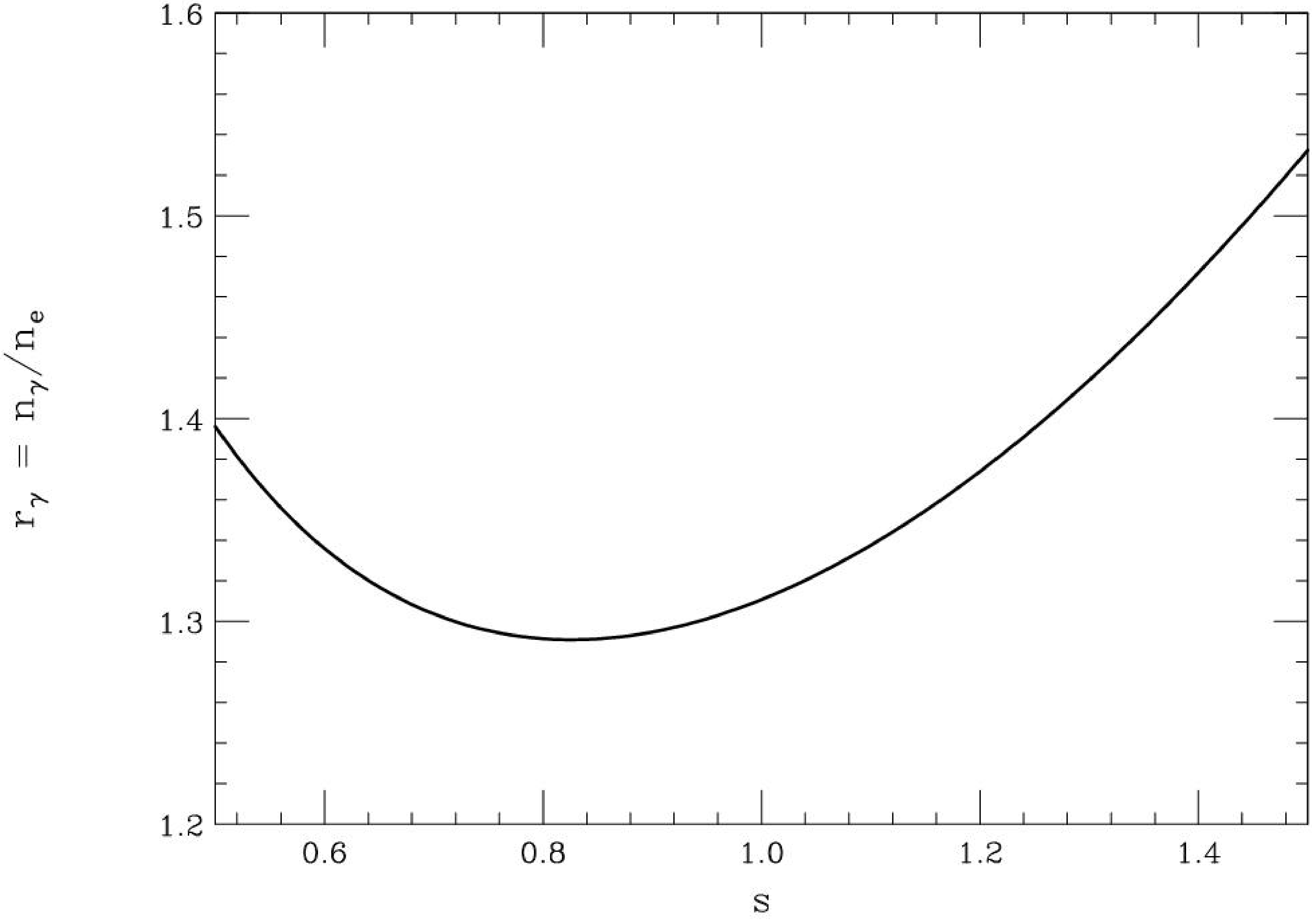
$$\lambda_1(s)$$

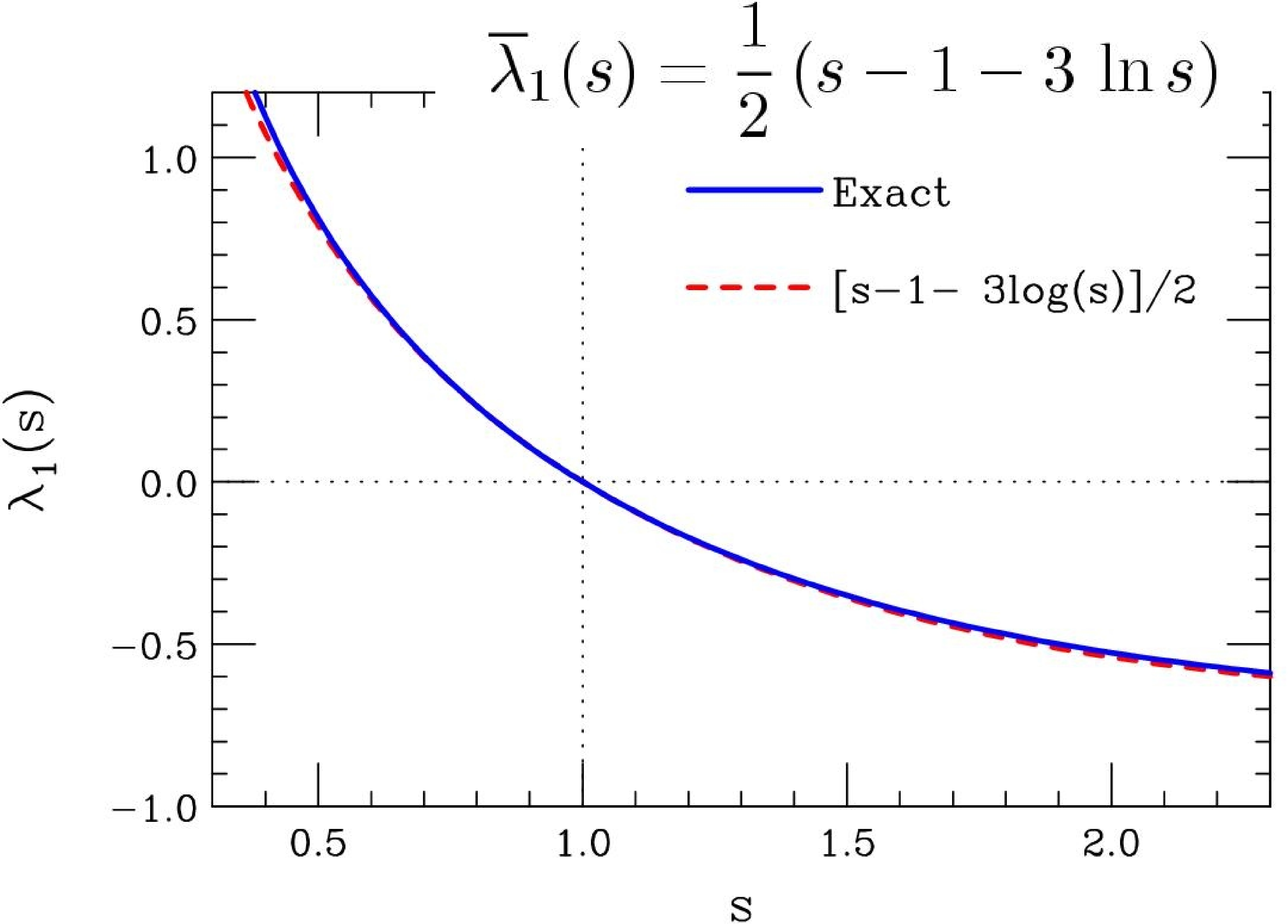
$$r_\gamma(s)$$

$$\begin{cases} n_e(E,t) &= K\; E^{-(s+1)}\; e^{\lambda_1(s)\,t} \\ n_\gamma(E,t) &= K\; r_\gamma(s)\; E^{-(s+1)}\; e^{\lambda_1(s)\,t} \end{cases}$$

$$\begin{cases} n_e(E,t) &= K\; E^{-(s+1)}\; e^{\lambda_2(s)\,t} \\ n_\gamma(E,t) &= K\; r_{\gamma 2}(s)\; E^{-(s+1)}\; e^{\lambda_2(s)\,t} \end{cases}$$

“Stability Ratio” for the Photon/Electron Ratio





$\overline{\lambda}_1(s) = \frac{1}{2}(s - 1 - 3\ln s)$
Exact
[s−1− 3log(s)]/2
1.0
0.5
0.0
−0.5
−1.0
0.5
1.0
1.5
2.0
λ1(s)
s

$$\lambda_{1,2}(s) = -\frac{1}{2}\left(A(s) + \sigma_0\right) \pm \frac{1}{2}\sqrt{\left(A(s) - \sigma_0\right)^2 + 4\,B(s)\,C(s)}$$

$$\begin{aligned} A(s) &= \int_0^1 dv\;\varphi(v)\;\left[1-(1-v)^s\right] \\ &= \left(\frac{4}{3}+2\,b\right)\left(\frac{\Gamma'(1+s)}{\Gamma(1+s)}+\gamma\right)+\frac{s\;(7+5\,s+12\,b\;(2+s))}{6\;(1+s)\;(2+s)} \\ B(s) &= 2\int_0^1 du\;u^s\;\psi(u) = \frac{2\;\left(14+11\,s+3\,s^2-6\,b\;(1+s)\right)}{3\;(1+s)\;(2+s)\;(3+s)} \\ C(s) &= \int_0^1 dv\;v^s\;\varphi(v) = \frac{8+7\,s+3\,s^2+6\,b\;(2+s)}{3\,s\;(2+3\,s+s^2)} \end{aligned}$$

$$\frac{dN_\gamma}{dE} = n_\gamma(E)$$

$$\frac{d\mathcal{E}_\gamma}{dE} = n_\gamma(E)\ E$$

$$\frac{d\mathcal{E}_\gamma}{d\ln E} = n_\gamma(E)\ E\ \frac{dE}{d\ln E}$$

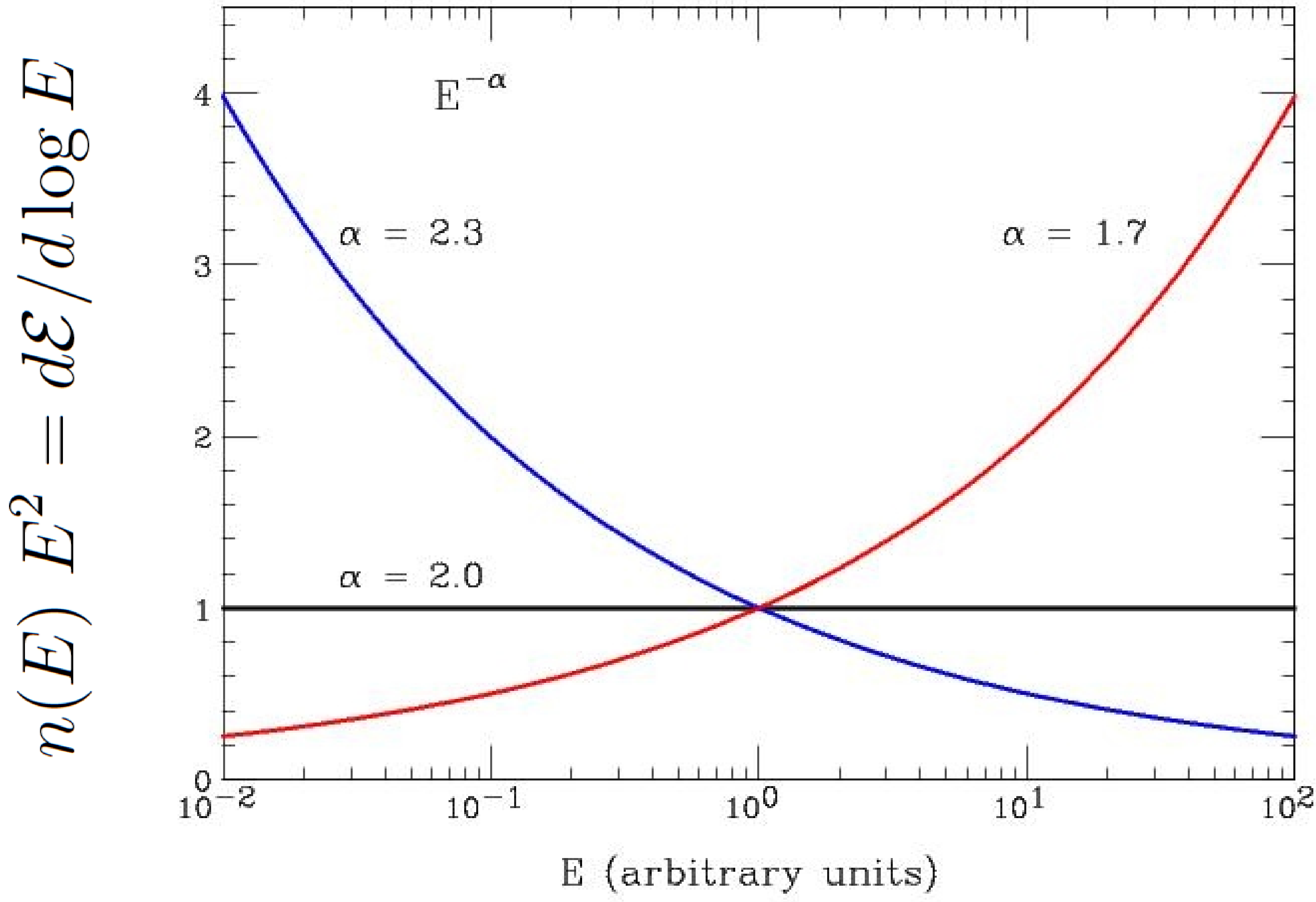
$$= n_\gamma(E)\ E^2$$

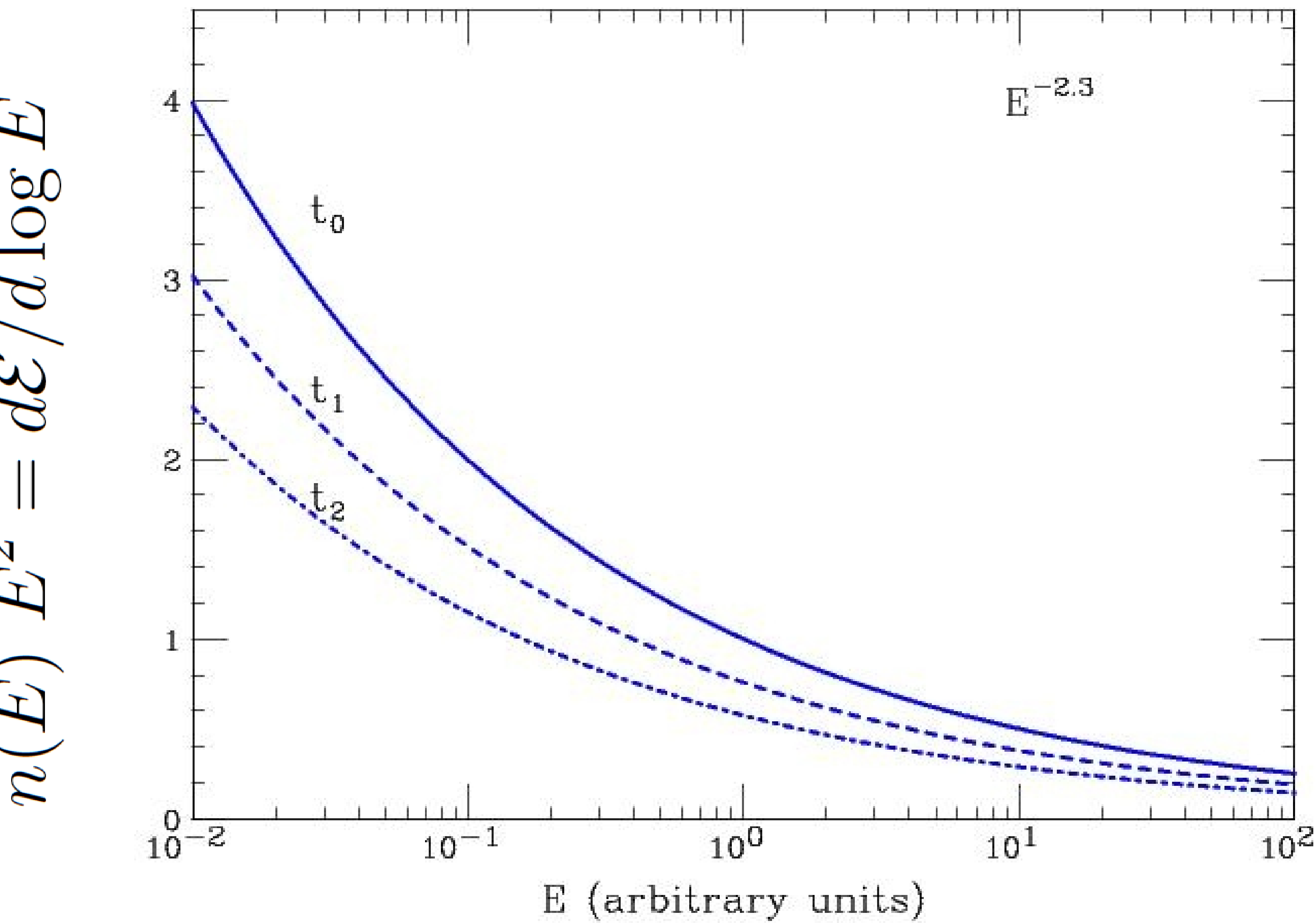
Concept of SPECTRAL ENERGY DISTRIBUTION:

$$\frac{d\mathcal{E}_\gamma}{d\log_{10} E} = n_\gamma(E)\ E^2\ \ln 10$$

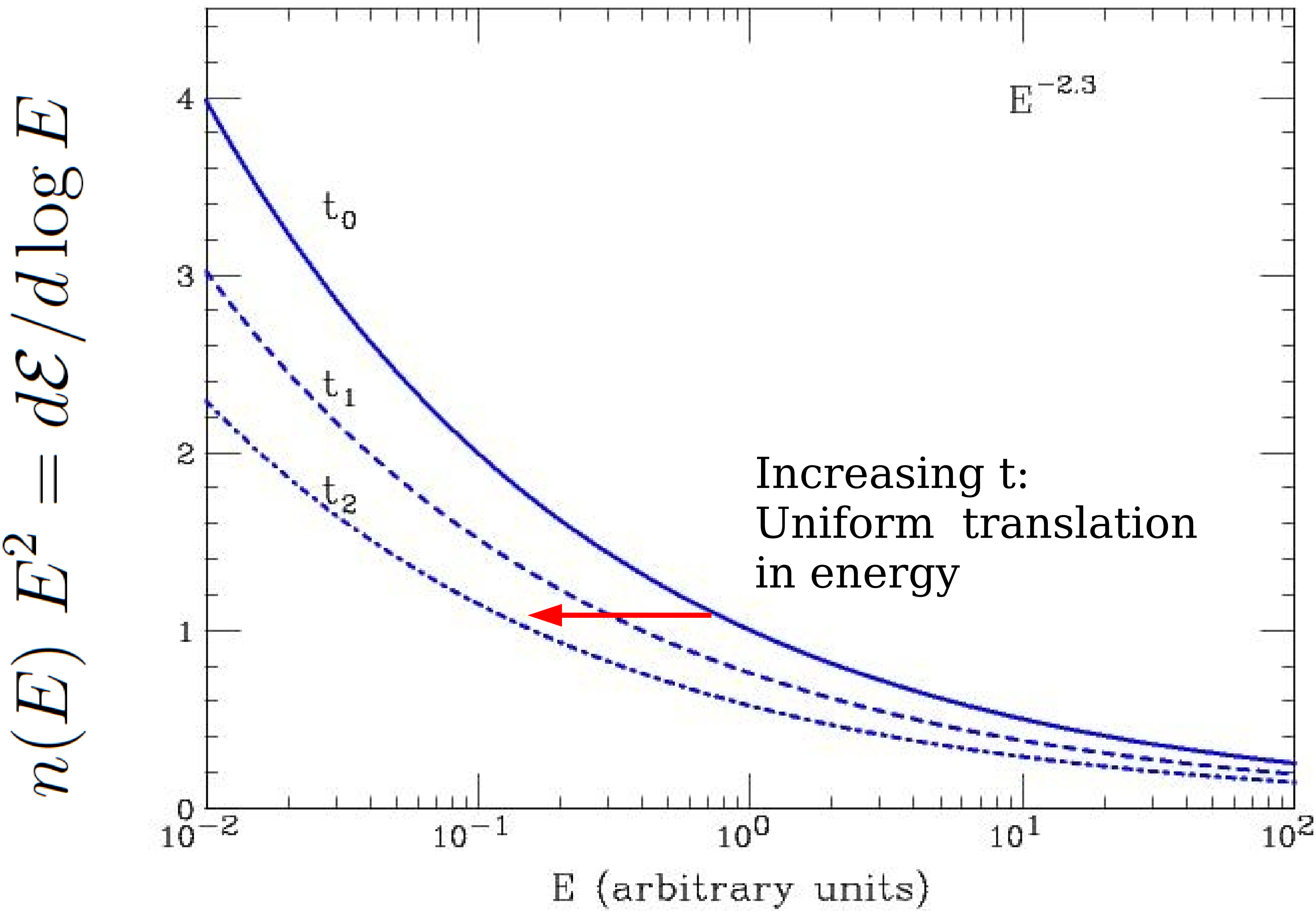
Amount of energy
carried by photons
per decade of energy

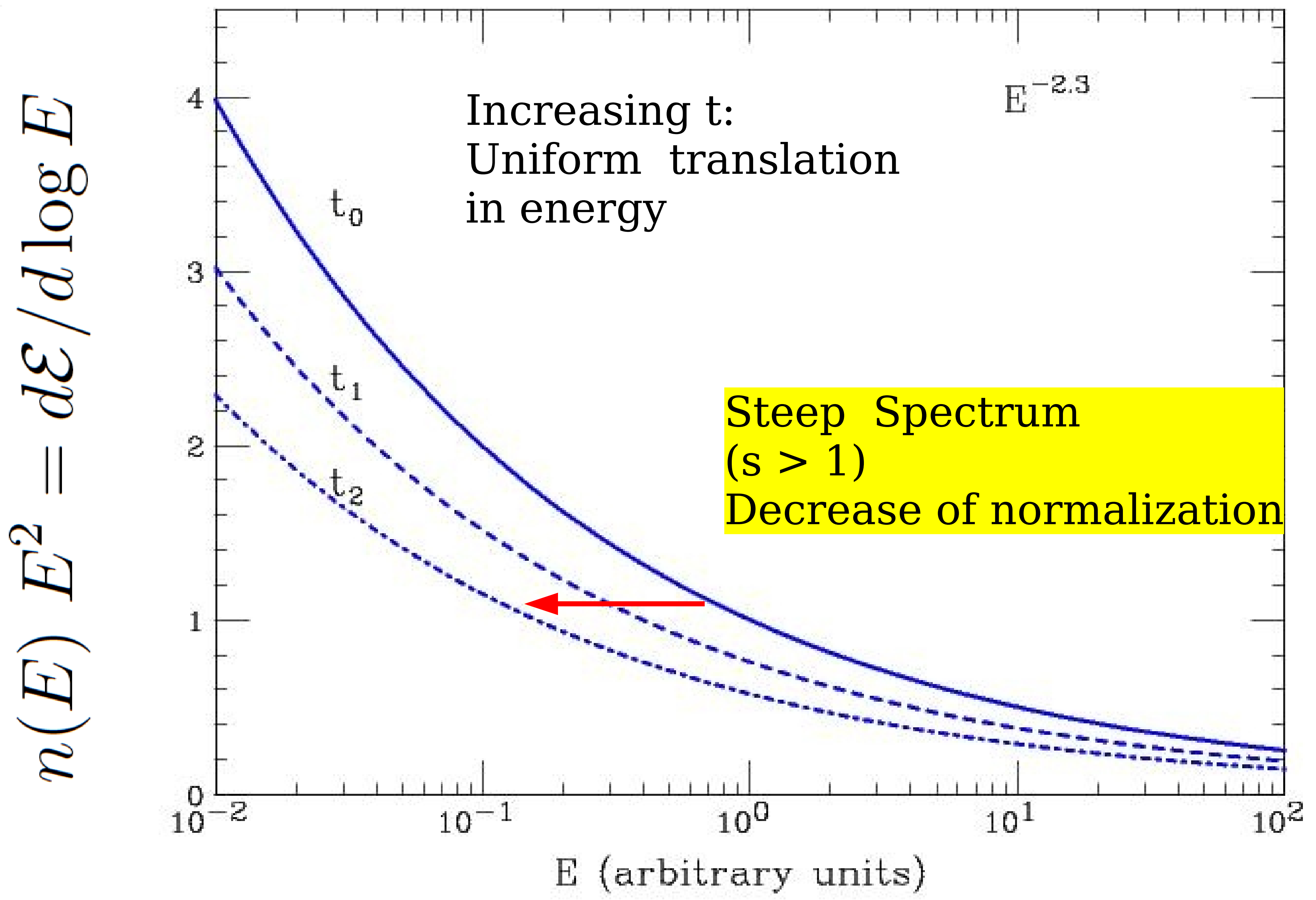
Power Law Solutions : Spectral Energy Distribution



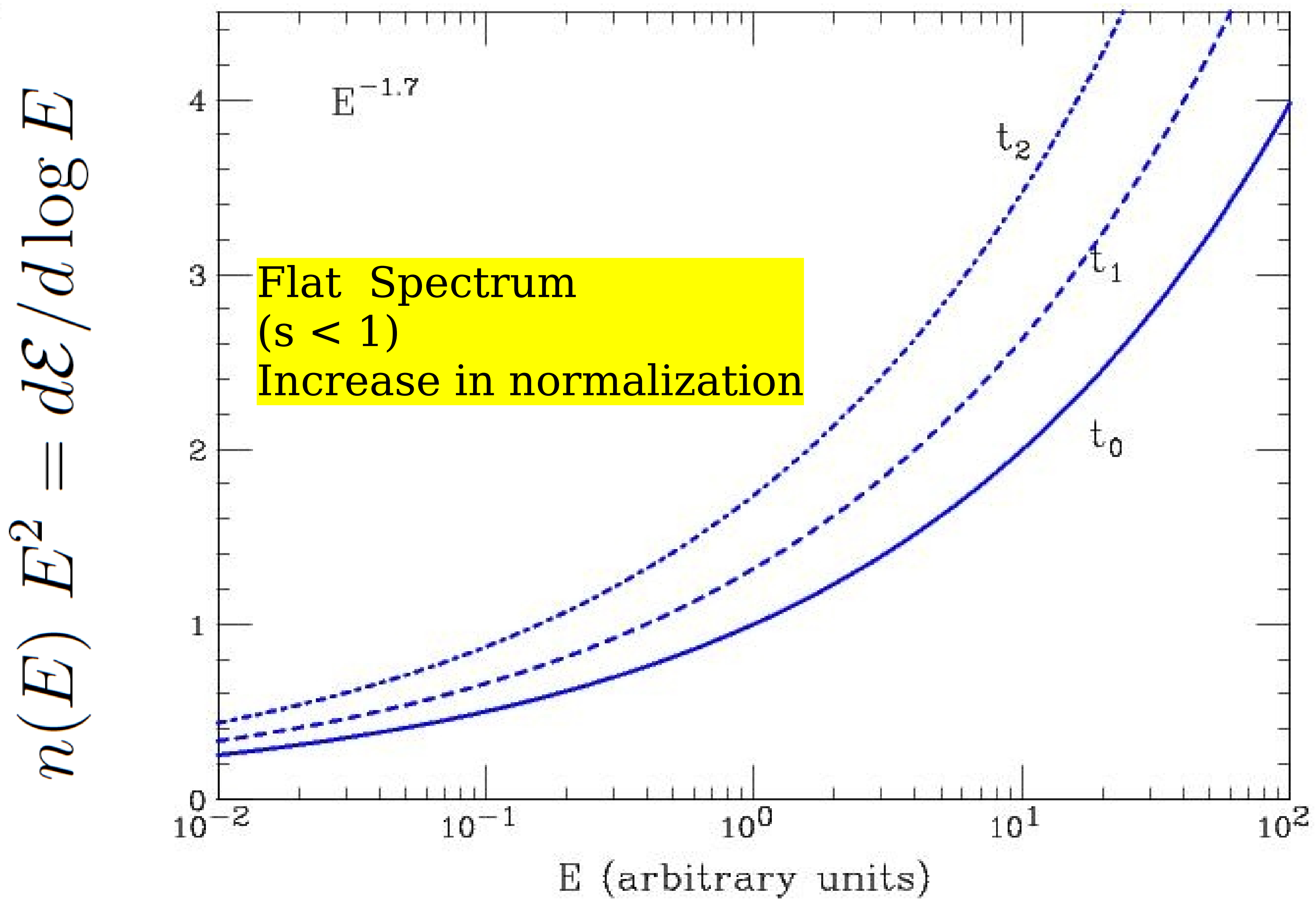


$n(E)\ E^2 = d\mathcal{E}/d\log E$
E (arbitrary units)
$E^{-2.3}$
t_0
t_1
t_2
0
1
2
3
4
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}


$n(E)\ E^2 = d\mathcal{E}/d\log E$
$E^{-2.3}$
t_0
t_1
t_2
Increasing t:
Uniform translation
in energy
E (arbitrary units)
0
1
2
3
4
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}



$n(E)\ E^2 = d\mathcal{E}/d\log E$
Increasing t:
Uniform translation
in energy
$E^{-2.3}$
t_0
t_1
t_2
Steep Spectrum
(s > 1)
Decrease of normalization
0
1
2
3
4
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
E (arbitrary units)



$n(E)\ E^2 = d\mathcal{E}/d\log E$
$E^{-1.7}$
Flat Spectrum
(s < 1)
Increase in normalization
t_2
t_1
t_0
0
1
2
3
4
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
E (arbitrary units)

“Elementary Solutions

$\lambda_1(s)$ $r_\gamma(s)$

Approximation A

$$\begin{cases} n_e(E,t) &= K\ E^{-(s+1)}\ e^{\lambda_1(s)\,t} \\ n_\gamma(E,t) &= K\ r_\gamma(s)\ E^{-(s+1)}\ e^{\lambda_1(s)\,t} \end{cases}$$

Approximation B

$E \gg \varepsilon$

= Approximation A

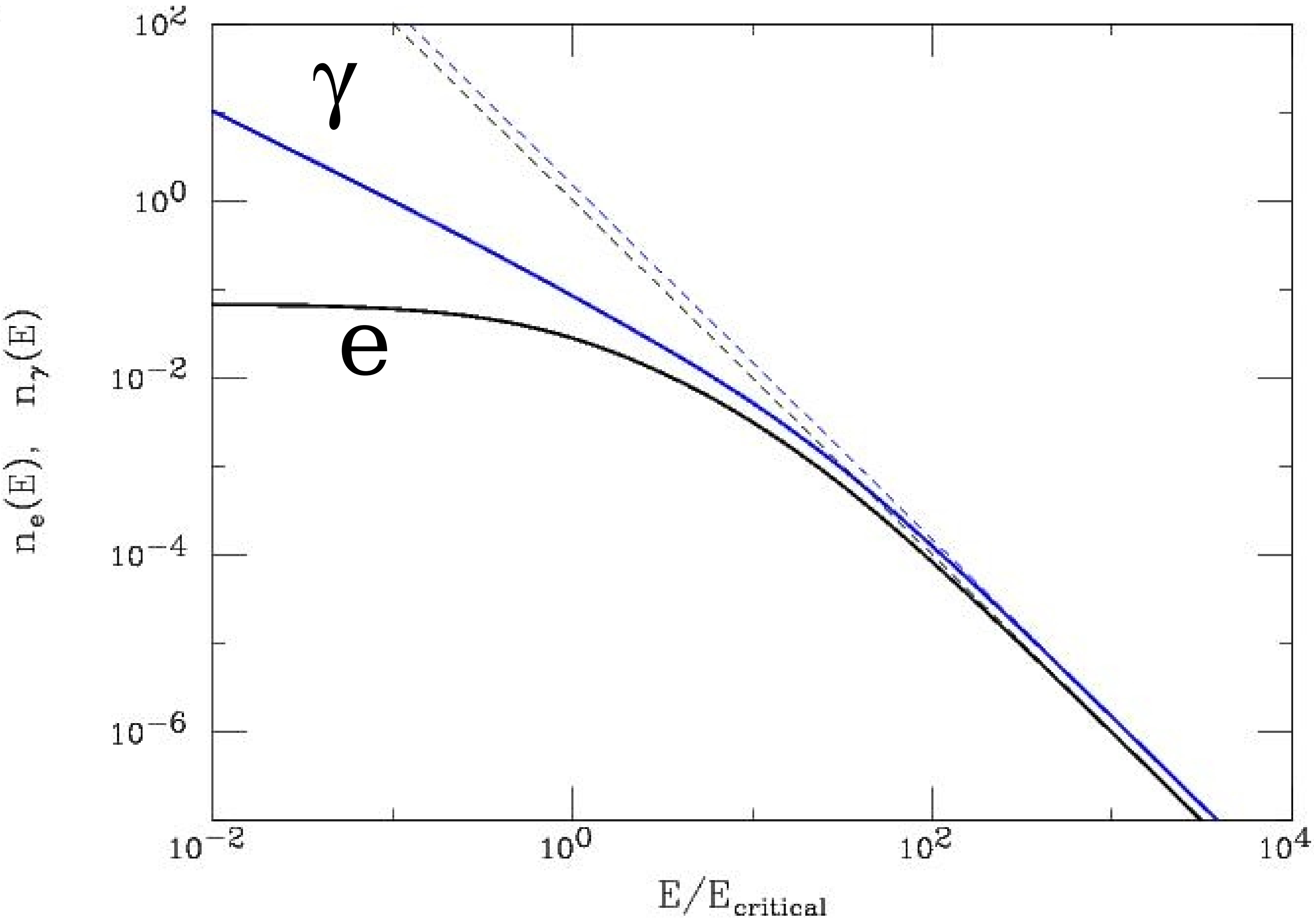
$$E/\varepsilon \to 0$$

$$n_e(E) \to \text{constant}$$

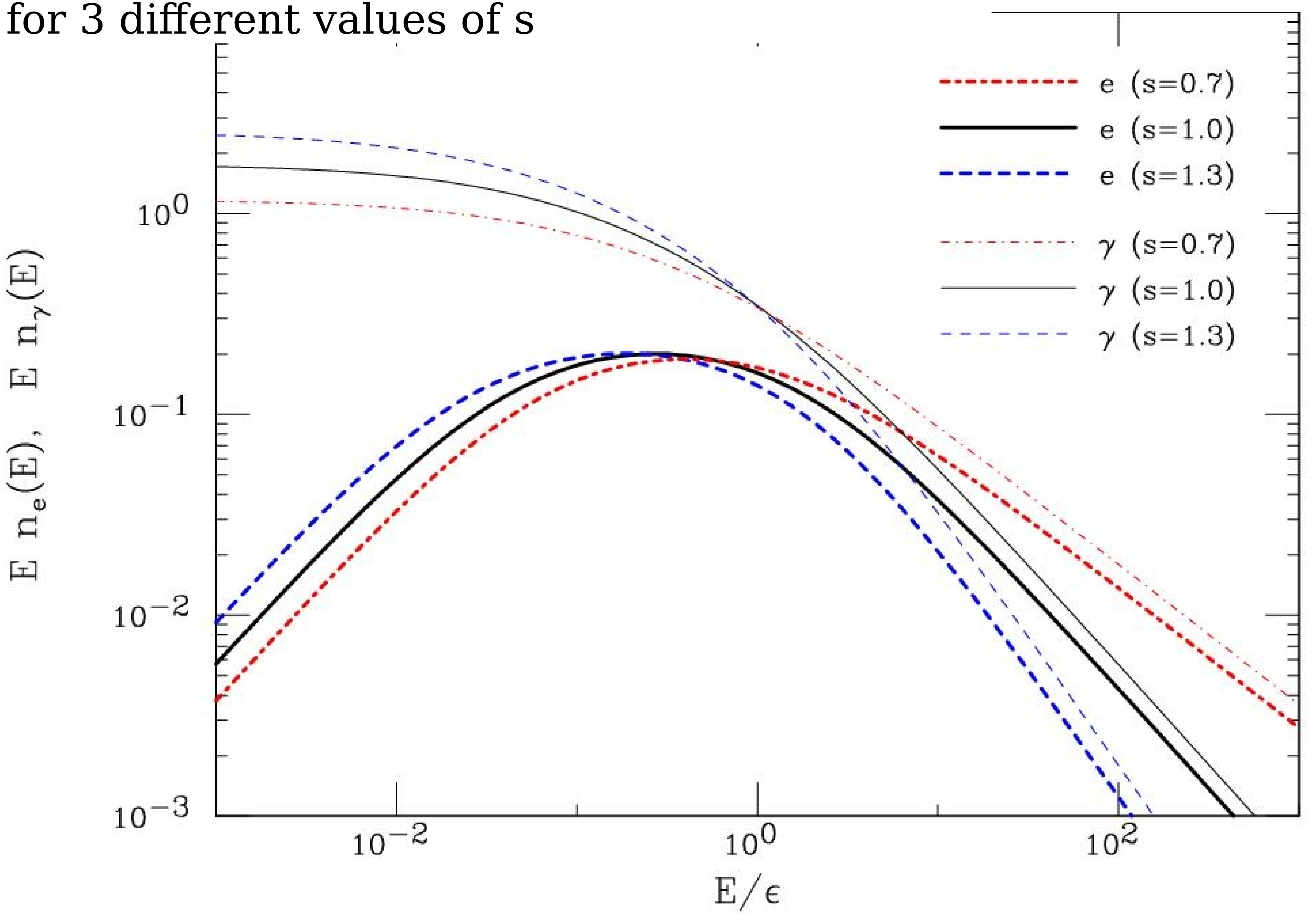
$$n_\gamma(E) \to E^{-1}$$

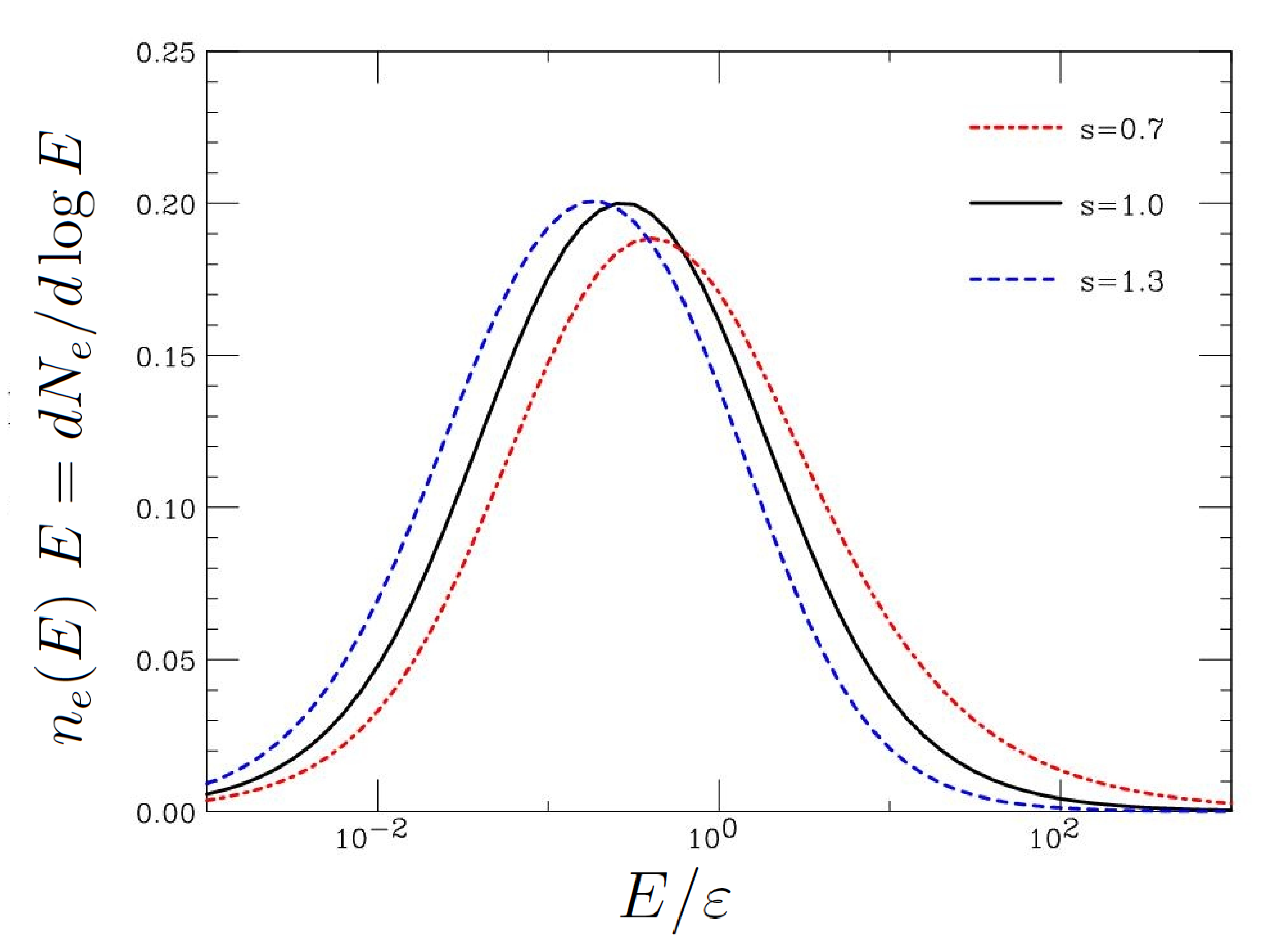
s=1

Approximation B "elementary solution"



Electron/photon spectra (elementary solution) for 3 different values of s





0.25
0.20
0.15
0.10
0.05
0.00
10^{-2}
10^{0}
10^{2}
s=0.7
s=1.0
s=1.3
$n_e(E)\ E = dN_e/d\log E$
E/ε

Solutions to the shower equations for the “real case”.

Initial Condition:

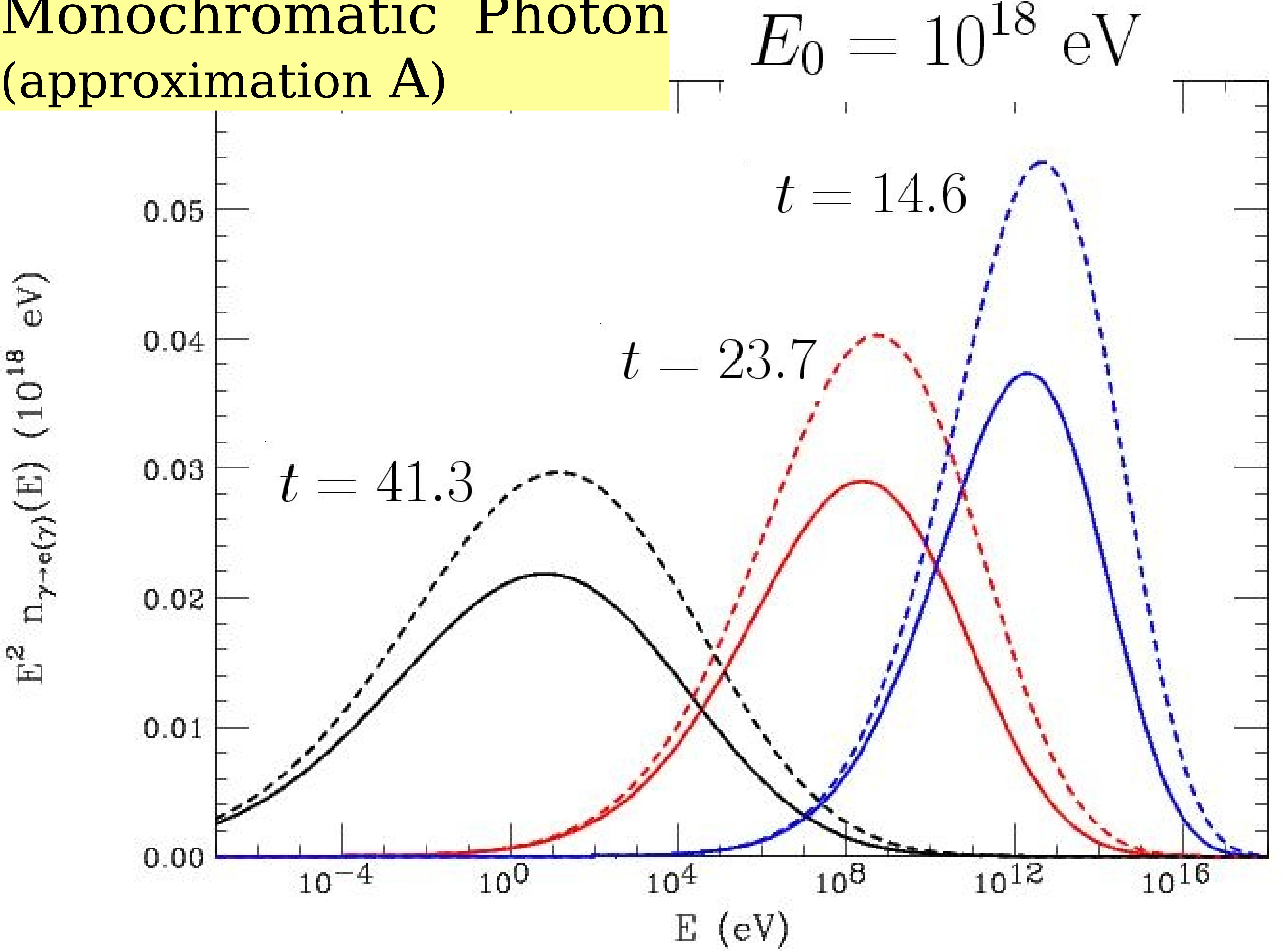
$$\begin{cases} n_e(E,0) & = & 0 \\ n_\gamma(E,0) & = & \delta[E-E_0] \end{cases}$$

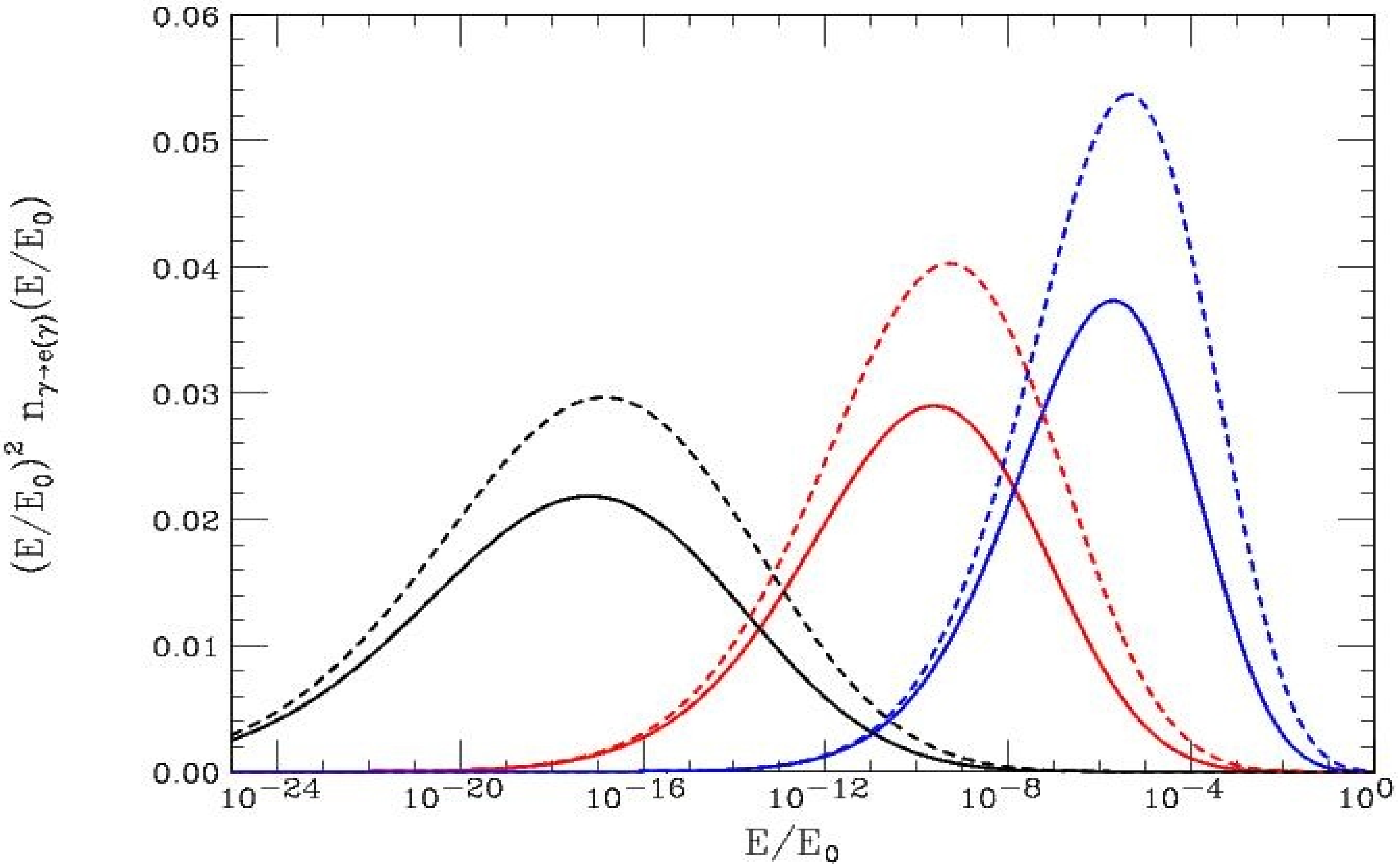
Photon of energy E_0

$$\begin{cases} n_e(E,0) & = & \delta[E-E_0] \\ n_\gamma(E,0) & = & 0 \end{cases}$$

Electron of energy E_0

Monochromatic Photon
(approximation A)





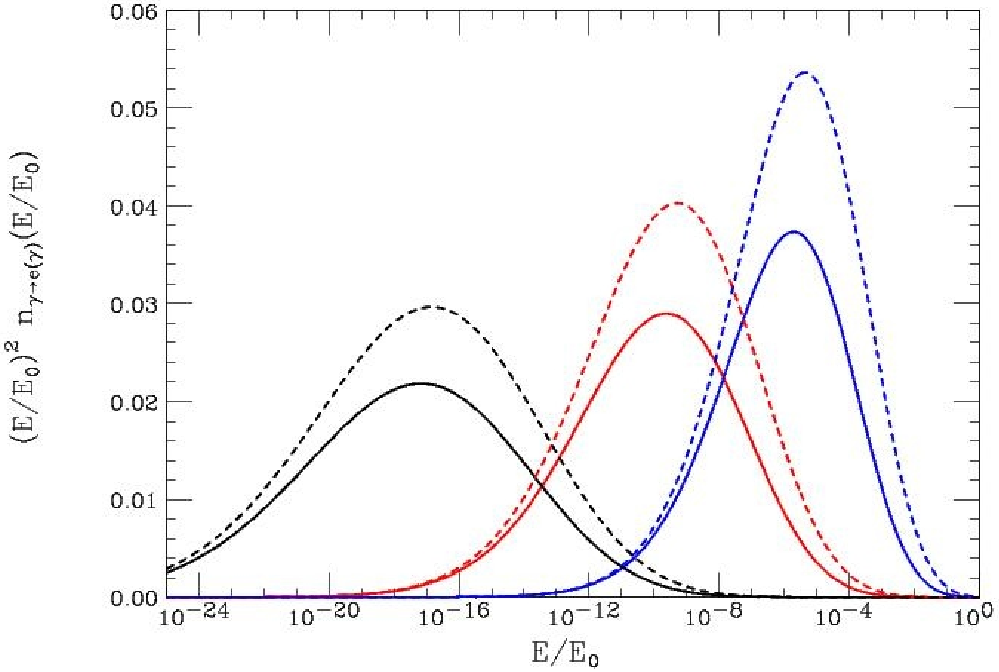
Solution valid for any initial energy

Function of E/E_0

$$n_\alpha(E_0, E, t) = \frac{1}{E_0}\, f_\alpha\left(\frac{E}{E_0}, t\right)$$

$\gamma \to e$ $\qquad e \to e$

$\gamma \to \gamma$ $\qquad e \to \gamma$



1. Energy Conservation

Area below
the curves constant
with t.

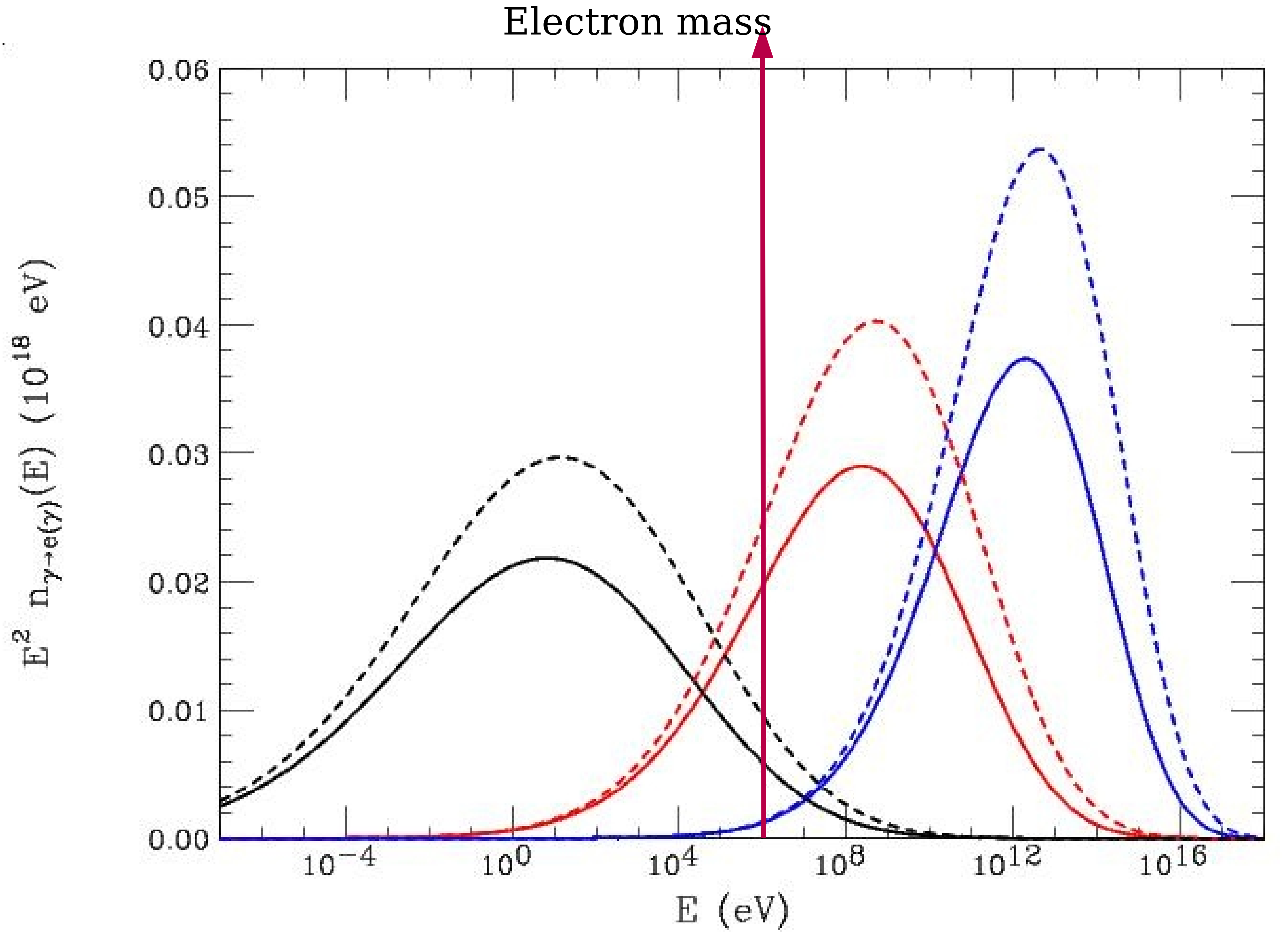
2. Electron and Photon
Spectra have very similar
shapes
The shapes are not
exactly identical
But the ratio gamma/e
is of order 1.3 , 1.4

Total ENERGY in a Shower

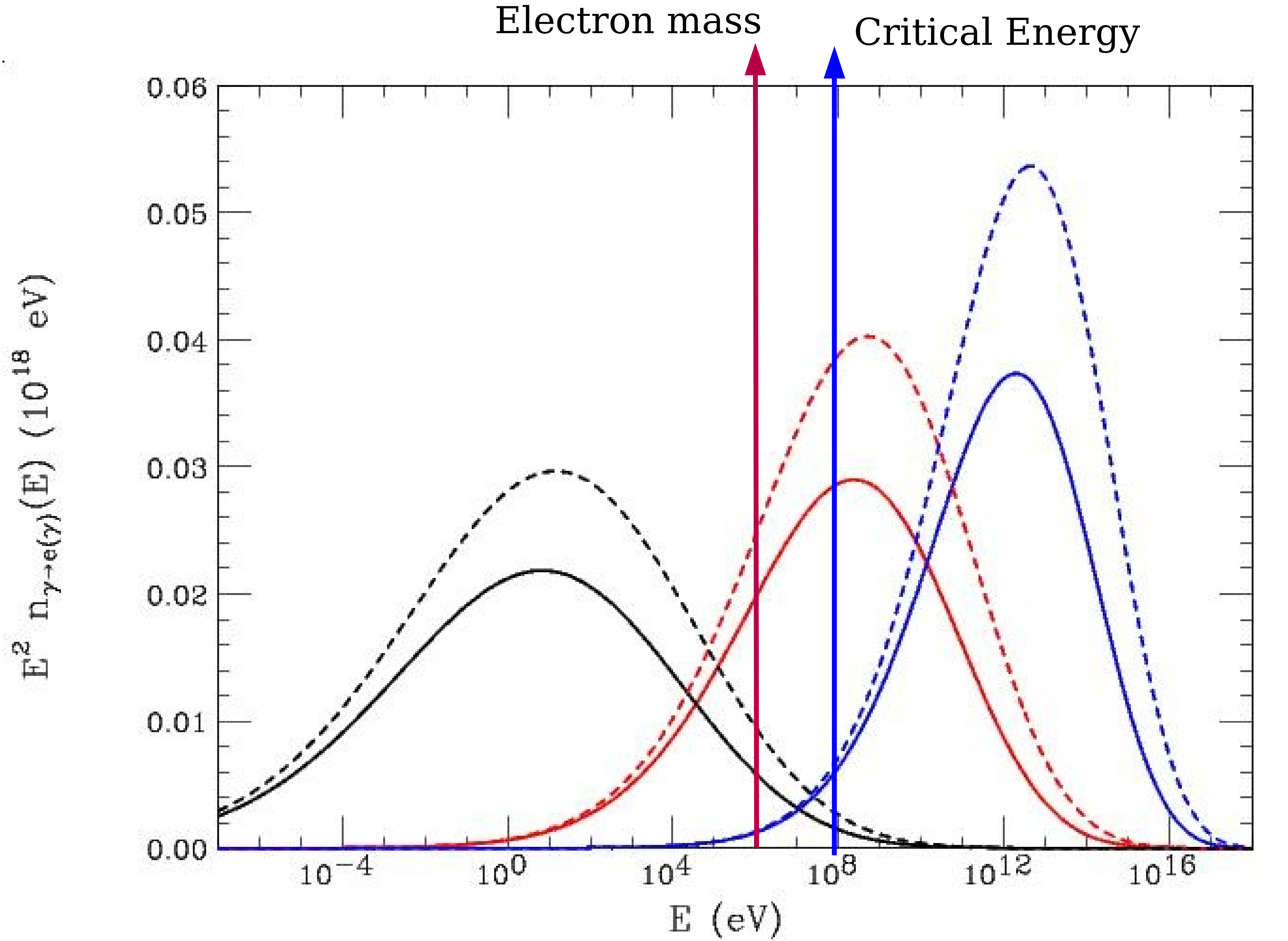
$$\mathcal{E}_{\rm shower}(t) = \mathcal{E}_{\rm electrons}(t) + \mathcal{E}_{\rm photons}(t)$$

$$\int_0^\infty dE\ E\ n_e(E,t) + \int_0^\infty dE\ E\ n_\gamma(E,t)$$

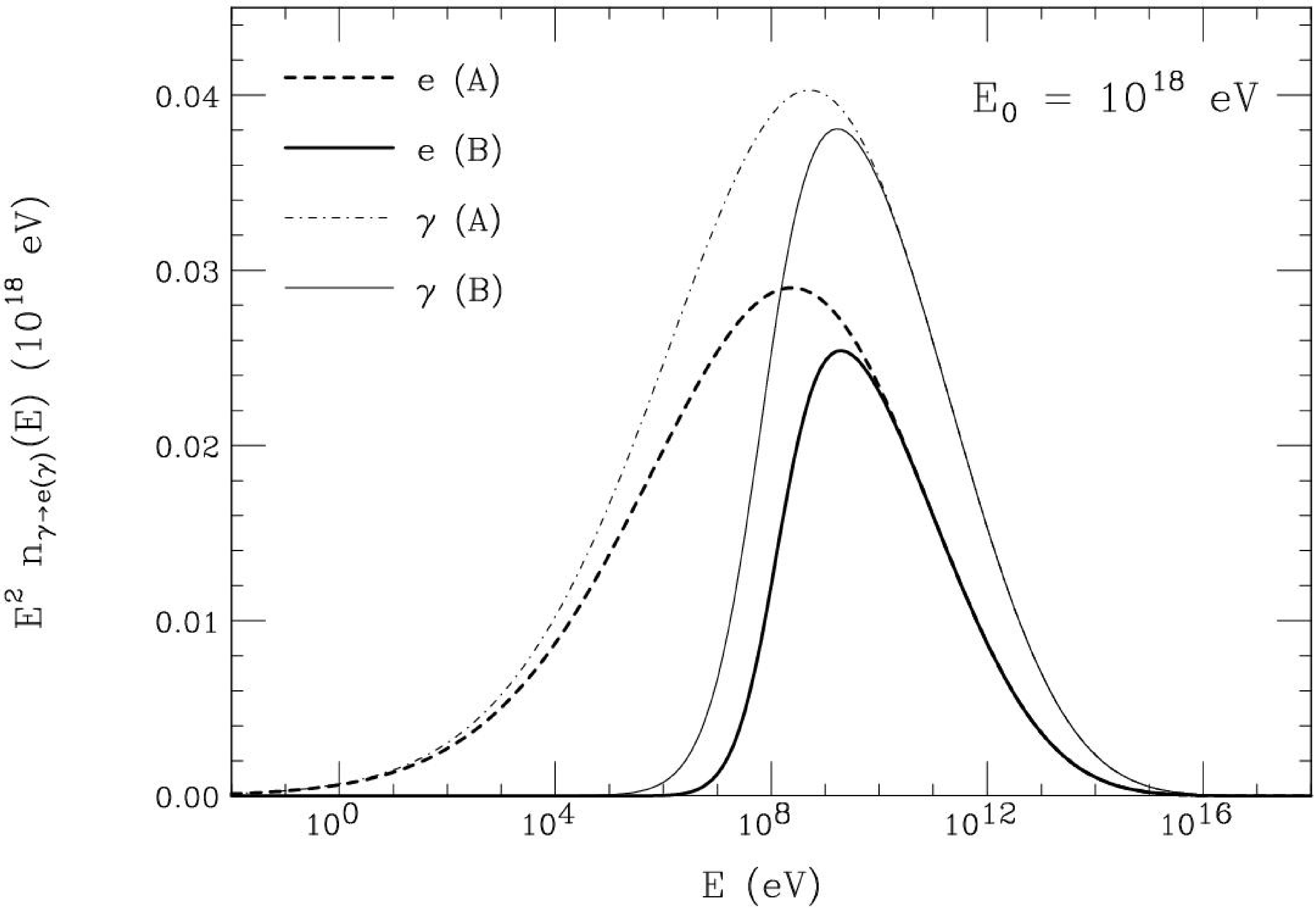
In Approximation A
the total Energy contained in Shower
is CONSTANT !

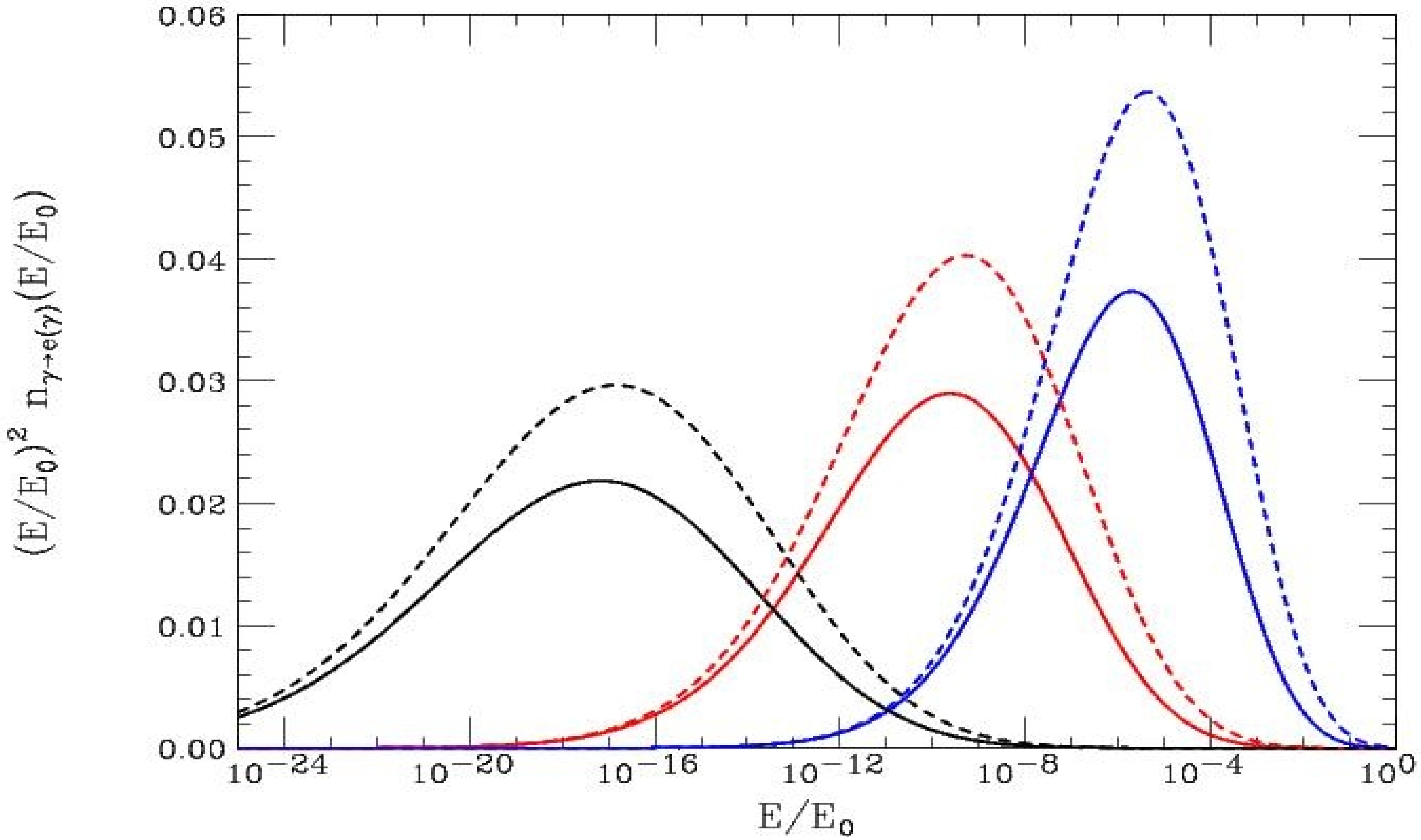


Electron mass
0.06
0.05
0.04
0.03
0.02
0.01
0.00
10^{-4}
10^{0}
10^{4}
10^{8}
10^{12}
10^{16}
E (eV)
$E^2\ n_{\gamma\to e(\gamma)}(E)\ (10^{18}\ eV)$


Electron mass
Critical Energy
0.06
0.05
0.04
0.03
0.02
0.01
0.00
$E^2\, n_{\gamma\to e(\gamma)}(E)$ (10^{18} eV)
10^{-4}
10^{0}
10^{4}
10^{8}
10^{12}
10^{16}
E (eV)

Monochromatic Photon. Approximation A,B

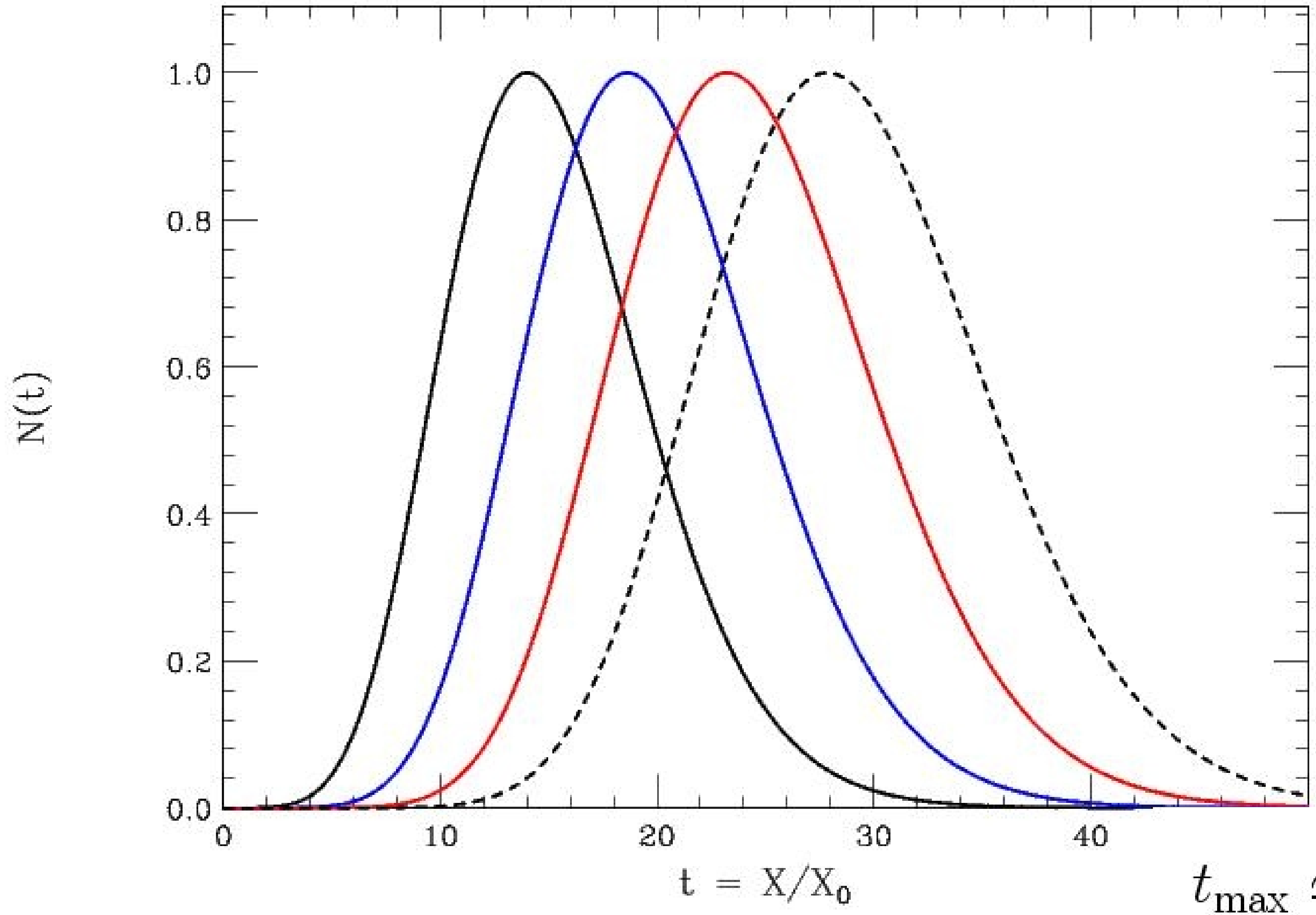




Choose one energy (any energy)
and study how the particle number varies
with t at that energy.

$$n_e(E, E_0, t) \propto \exp\left[t\left(1 - \frac{3}{2}\log\left(\frac{3t}{t + 2\,\ln(E_0/E)}\right)\right)\right]$$

(good approximation)



$$t_{\max} \simeq \ln\left(\frac{E_0}{E}\right)$$

$$n(E) = E^{-\alpha}$$

Power Law

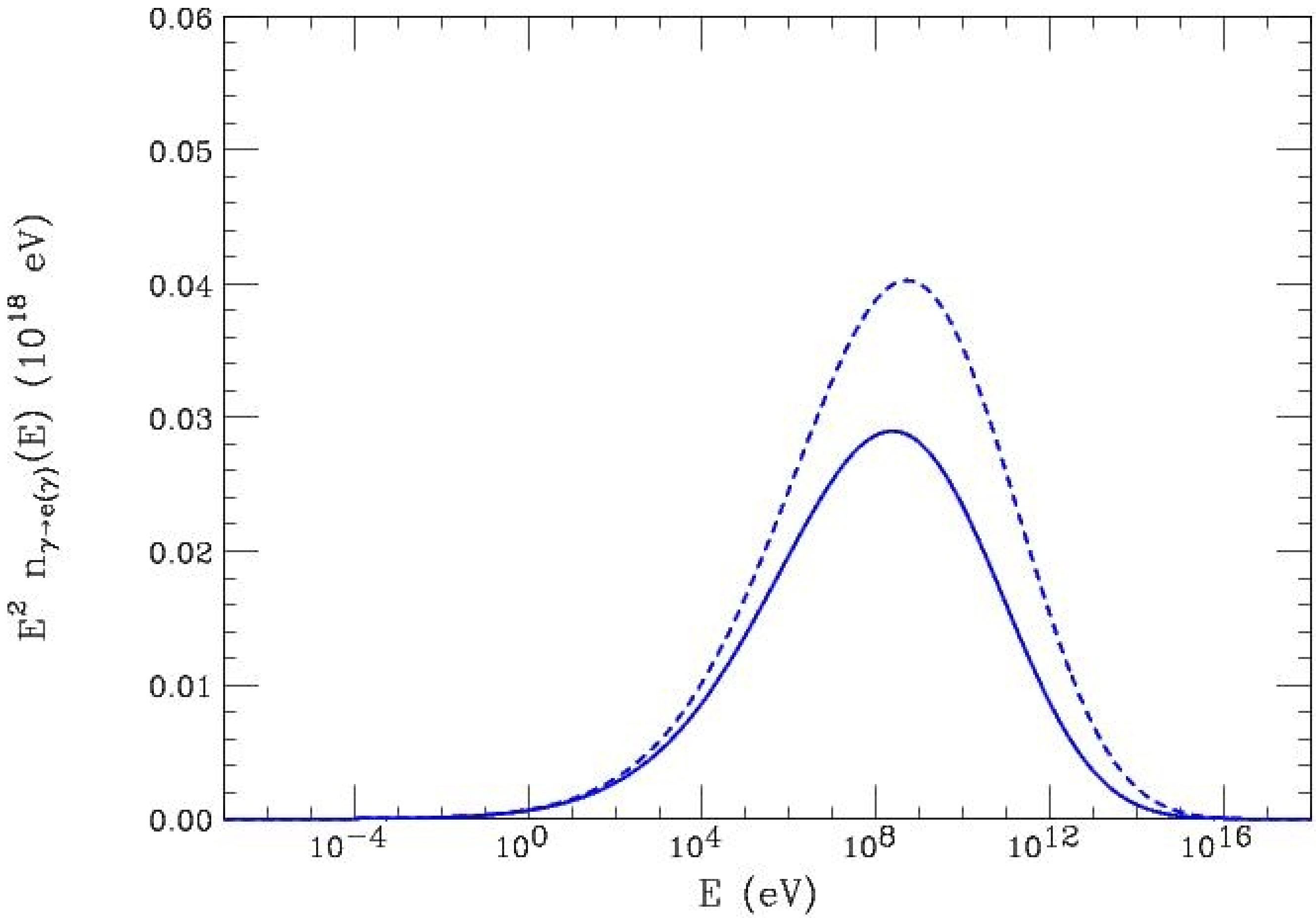
$$\alpha = -\left(\frac{E}{n}\right) \frac{dn(E)}{dE}$$

Slope

$n(E)$ Arbitrary shape

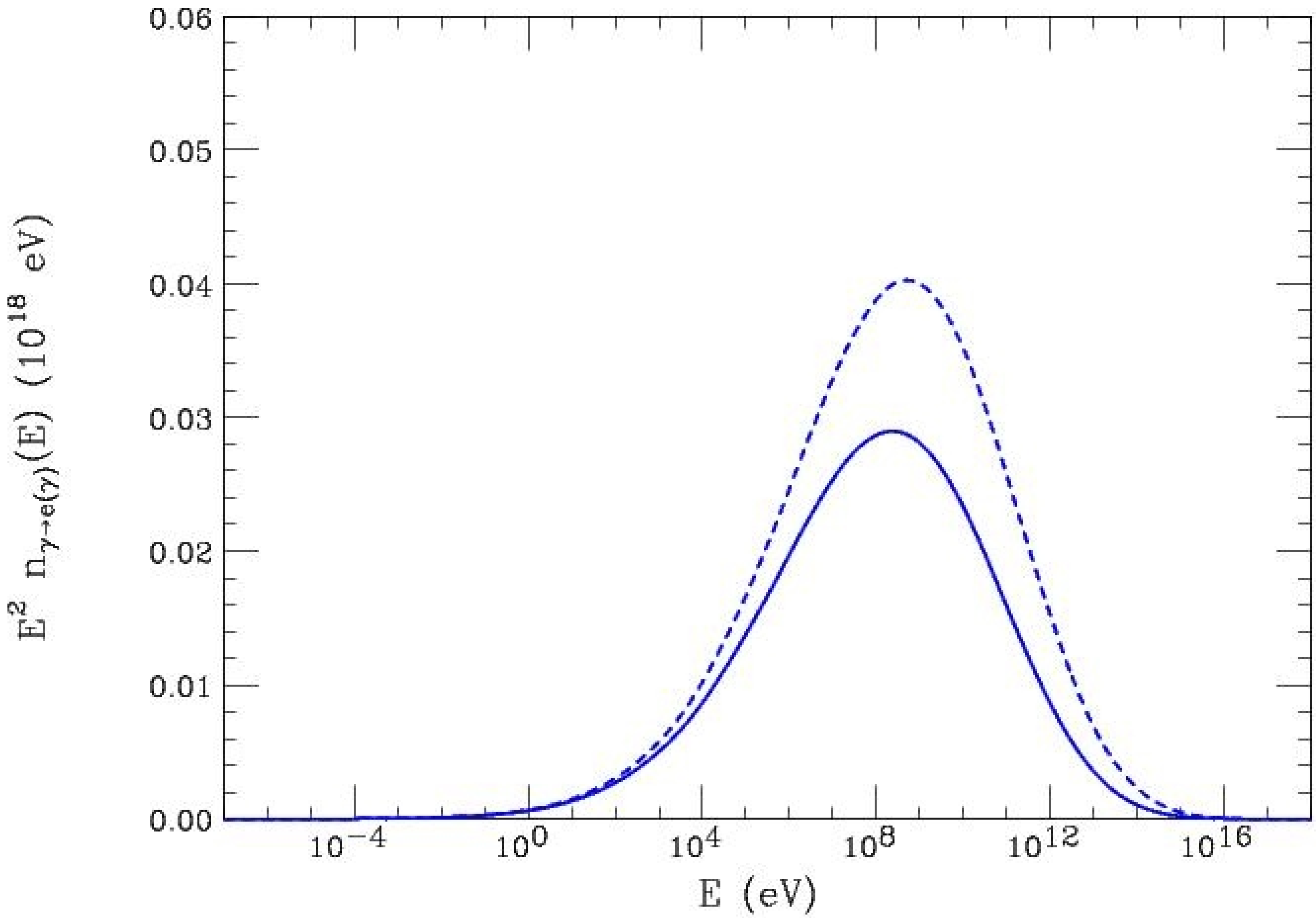
$$\alpha(E) = s(E) + 1 = -\left(\frac{E}{n}\right) \frac{dn(E)}{dE}$$

"Local (energy dependent) Slope"



Consider the shape of the spectra at a fixed t
It is a function of E/E0 and t.

$s(E/E_0, t)$ Local slope



$$s(E/E_0, t)$$

Consider the shape of the spectra at a fixed t
It is a function of E/E0 and t.

QUESTION : At what energy in this graph s(E) = 1 ?

t-slope and E-slope are connected

$$\lambda = \frac{1}{N(t)} \frac{dN(t)}{dt}$$

Integral Electron Spectrum Evolution

Can deduce the AGE (and spectral shape)

$$s = \lambda_1^{-1}(\lambda) = \lambda_1^{-1}\left(\frac{1}{N(t)} \frac{dN(t)}{dt}\right)$$

$$n_e(E) \sim n_\gamma(E) \sim E^{-(s+1)}$$

So ....

How can we obtain these results from the shower equations ?

So ....

How can we obtain
The solution from the
shower equations ?

We know how to solve for an initial condition that is a power law

$$n_e(E,0) = \delta[E - E_0]$$

Write initial condition
as a superposition of power law component
Inverse Mellin transform

$$f(E) = \frac{1}{2\pi i} \int_C ds \; E^{-(s+1)} \; M_f(s)$$

$$M_f(s) = \int_0^\infty dE \; E^s \; f(E)$$

$$n_e(E,0) = \frac{1}{2\pi i} \int_C ds \; E_0^s \; E^{-(s+1)}$$

The parameter s takes complex values

$$n_e(E,0) = \delta[E - E_0]$$

Write initial condition
as a superposition of power law
component
Inverse Mellin transform

$$f(E) = \frac{1}{2\pi i} \int_C ds \; E^{-(s+1)} \; M_f(s)$$

$$M_f(s) = \int_0^\infty dE \; E^s \; f(E)$$

$$n_e(E,0) = \frac{1}{2\,\pi\, i} \int_C ds \; E_0^s \; E^{-(s+1)}$$

Depth Evolution

$$n_e(E,t) \simeq \frac{1}{2\,\pi\, i} \int_C ds \; E_0^s \; E^{-(s+1)} \; e^{\lambda_1(s)\, t}$$

For a given E_0, E, t what is the parameter s that dominate ?

SADDLE point Approximation

$$n_e(E,t) \simeq \frac{1}{2\pi i} \int_C ds \; E_0^s \; E^{-(s+1)} \; e^{\lambda_1(s)\,t}$$

$$\frac{d}{ds}\left[\left(\frac{E}{E_0}\right)^{-s} \; e^{\lambda_1(s)\,t}\right] = 0$$

$$\lambda'(s)\,t + \ln\left(\frac{E_0}{E}\right) = 0$$

Solution of this equation

For a given E_0, E, t

what is the parameter s that dominate ?

$$n_e(E,t) \simeq \frac{1}{2\,\pi\,i} \int_C ds\; E_0^s\; E^{-(s+1)}\; e^{\lambda_1(s)\,t}$$

$$\frac{d}{ds}\left[\left(\frac{E}{E_0}\right)^{-s} \; e^{\lambda_1(s)\,t}\right] = 0$$

$$\lambda'(s)\,t + \ln\left(\frac{E_0}{E}\right) = 0$$

Solution of this equation

$$\overline{\lambda}_1(s) = \frac{1}{2}\,(s - 1 - 3\,\ln s)$$

$$s \simeq \overline{s}\left(\frac{E}{E_0}, t\right) = \frac{3\,t}{t - 2\,\ln(E/E_0)}$$

Age and Longitudinal Development

$$\frac{dN_e(t)}{dt} = \lambda_1(s)\ N_e(t)$$

Age and Longitudinal Developmen t

$$\frac{dN_e(t)}{dt} = \lambda_1(s)\ N_e(t)$$

S = $\dfrac{3\,t}{t + 2\,t_{\max}}$

$$\overline{\lambda}_1(s) = \frac{1}{2}\,(s - 1 - 3\,\ln s)$$

Age and Longitudinal Developmen t

$$\frac{dN_e(t)}{dt} = \lambda_1(s)\ N_e(t)$$

$$s = \frac{3\,t}{t + 2\,t_{\max}}$$

$$\overline{\lambda}_1(s) = \frac{1}{2}\,(s - 1 - 3\,\ln s)$$

$$= \frac{1}{2}\left[\frac{3t}{t + 2\,t_{\max}} - 1 - 3\,\log\left(\frac{3t}{t + 2\,t_{\max}}\right)\right]\ N(t)$$

Differential Equation

Differential Equation

$$\frac{dN_e(t)}{dt} = \lambda_1(s)\ N_e(t)$$

$$= \frac{1}{2}\left[\frac{3t}{t+2\,t_{\max}} - 1 - 3\,\log\left(\frac{3t}{t+2\,t_{\max}}\right)\right]\ N(t)$$

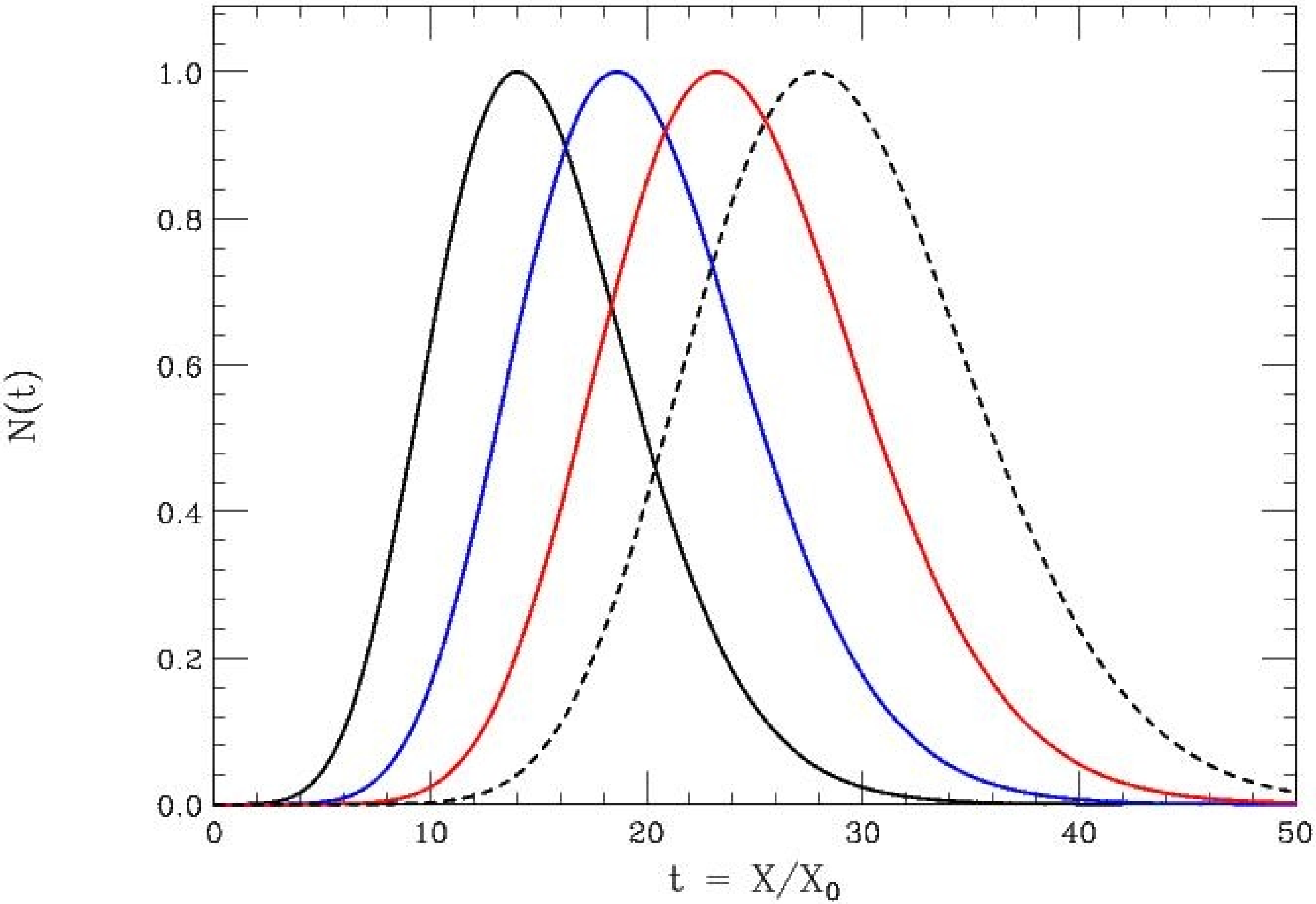
Boundary Condition

$$N(t_{\max}) = N_{\max}$$

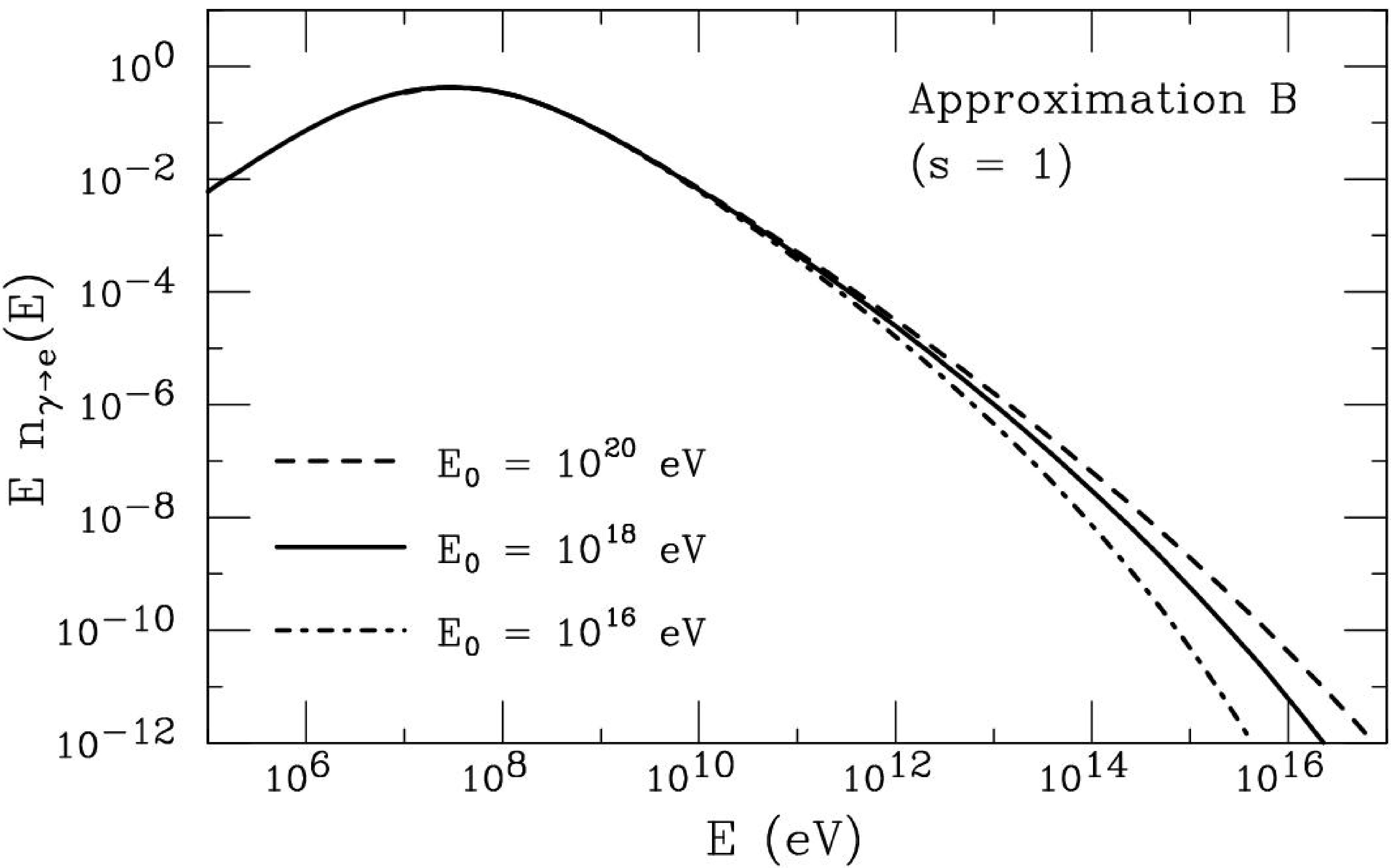
Solution : Greisen Profile

$$N_e(t) = N_{\max}\ e^{-t_{\max}}\ \exp\left[t\left(1 - \frac{3}{2}\log\left(\frac{3t}{t+2\,t_{\max}}\right)\right)\right]$$

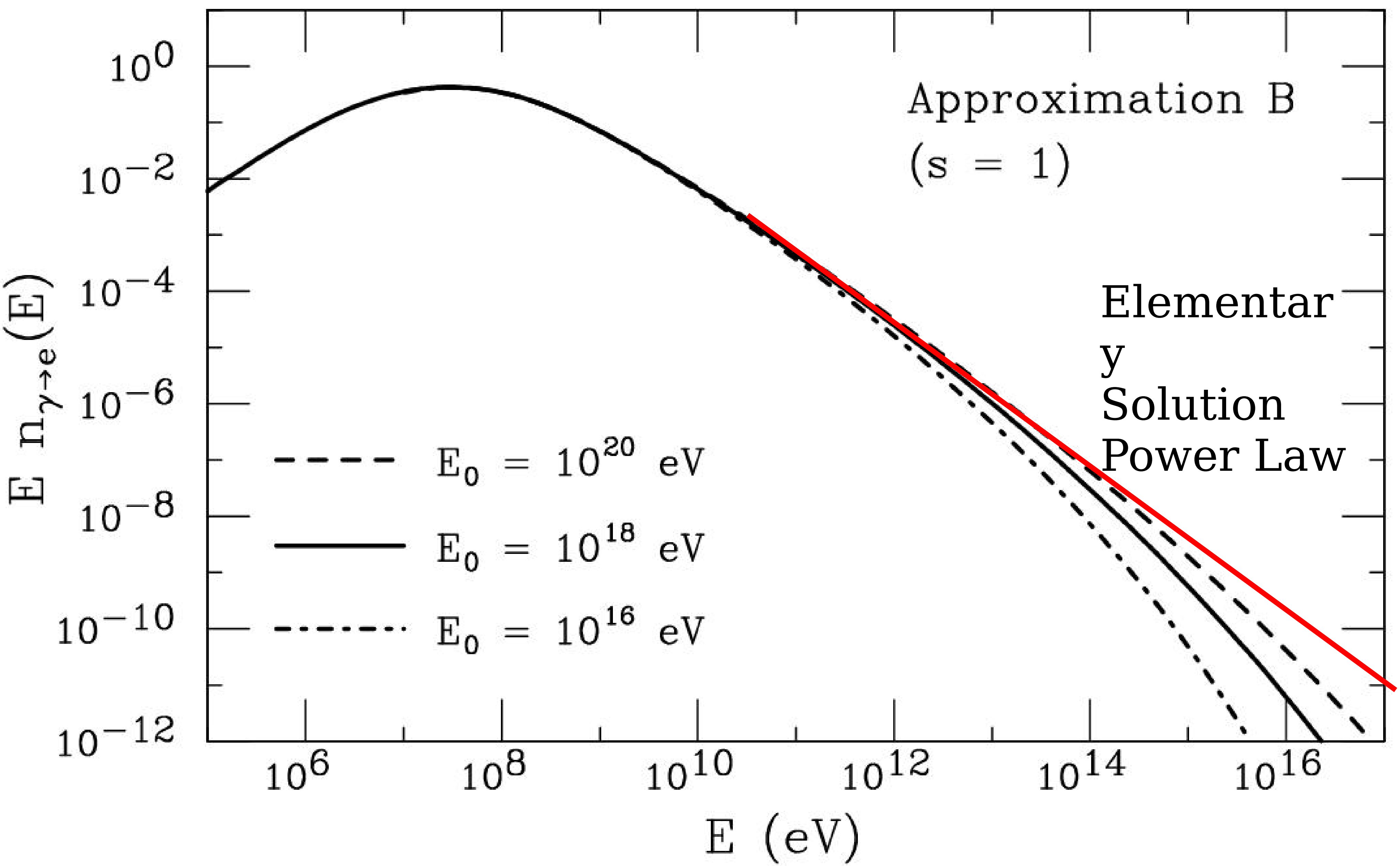
$$N_{\mathrm{Greisen}}(E_0, t) = \frac{0.31}{\sqrt{\ln(E_0/\varepsilon)}} \exp\left[t\left(1 - \frac{3}{2}\log\left(\frac{3t}{t + 2\ln(E_0/\varepsilon)}\right)\right)\right]$$

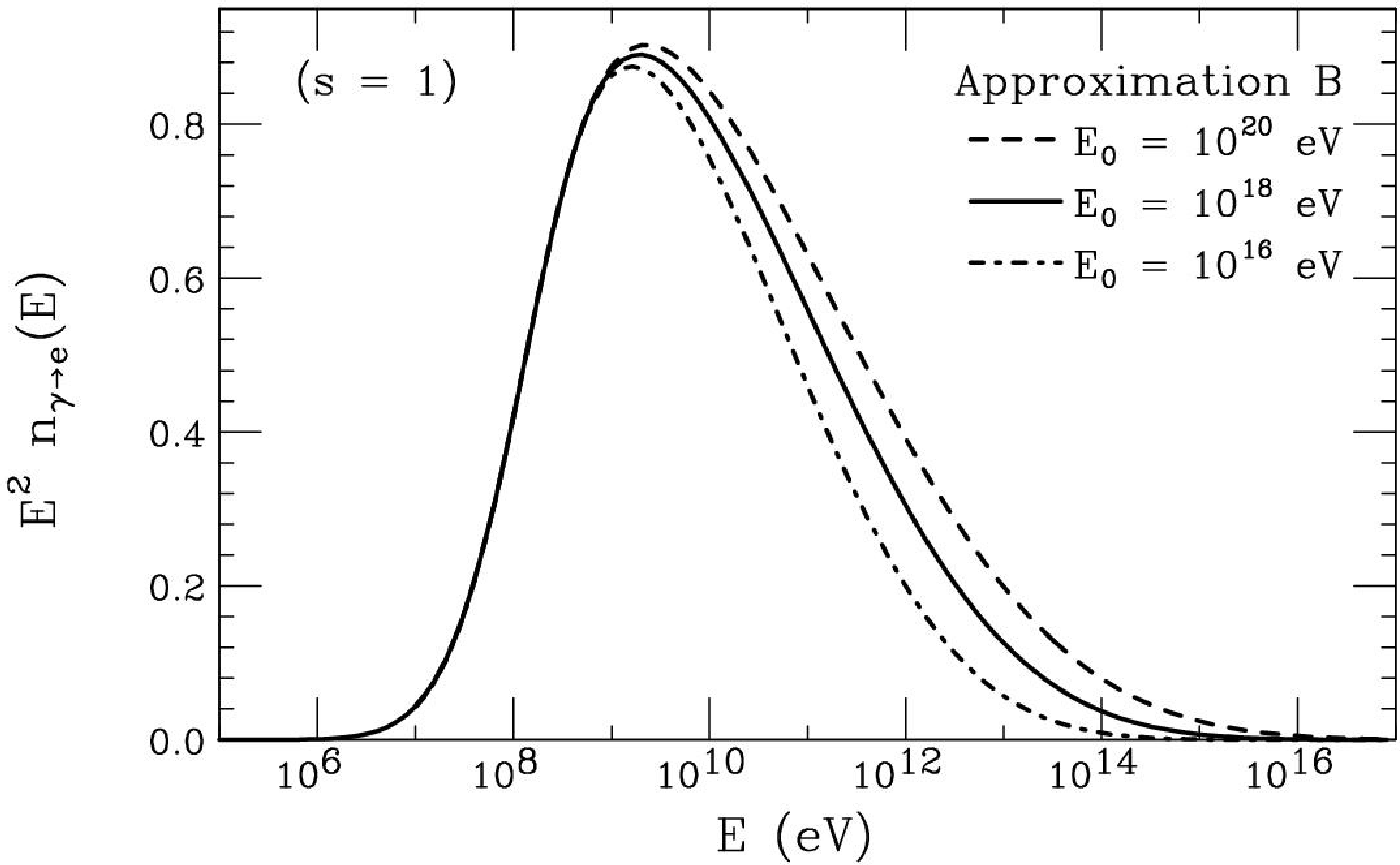


Different Energy : Same Age (Shower Maximum)



Different Energy : Same Age (Shower Maximum)





(s = 1)
Approximation B
$E_0 = 10^{20}$ eV
$E_0 = 10^{18}$ eV
$E_0 = 10^{16}$ eV
0.0
0.2
0.4
0.6
0.8
10^6
10^8
10^{10}
10^{12}
10^{14}
10^{16}
E (eV)
$E^2\ n_{\gamma\to e}(E)$

ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT

Astroparticle Physics 24 (2006) 421–437

Astroparticle Physics

www.elsevier.com/locate/astropart

Universality of electron distributions in high-energy air showers—Description of Cherenkov light production

F. Nerling [a,*], J. Blümer [a,b], R. Engel [a], M. Risse [a]

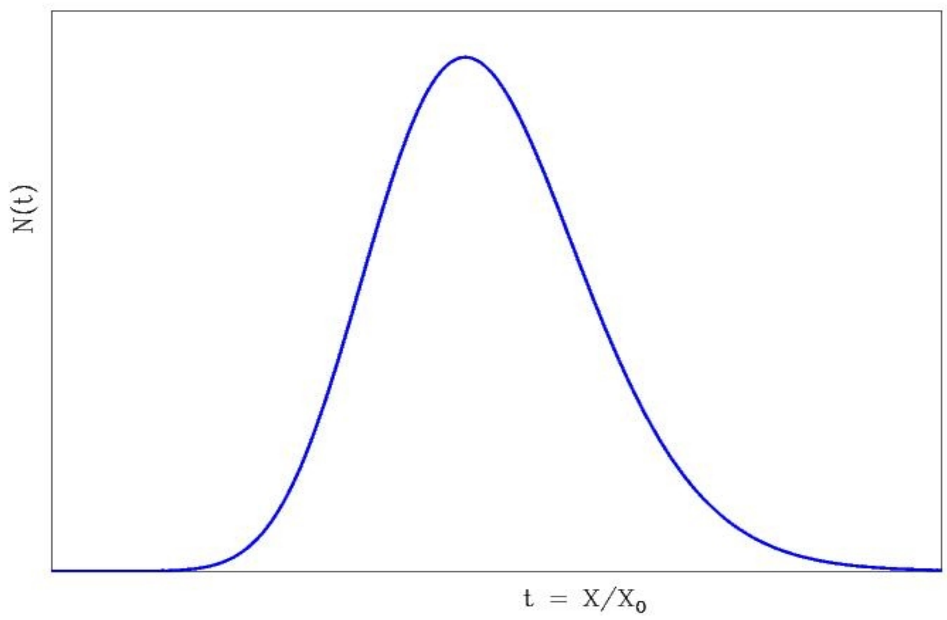
Abstract

The shower simulation code CORSIKA has been used to investigate the electron energy and angular distributions in high-energy showers. Based on the universality of both distributions, we develop an analytical description of Cherenkov light emission in extensive air showers, which provides the total number and angular distribution of photons. The parameterisation can be used e.g. to calculate the contribution of direct and scattered Cherenkov light to shower profiles measured with the air fluorescence technique.

Earlier results

M. Giller et al., J. Phys. G: Nucl. Part. Phys. 30 (2004) 97;
M. Giller, in: Proc. 28th Int. Cos. Ray Conf., Tsukuba, Japan, vol. 2, 2003, p. 619. Note: The set of parameters

Concept of : Shower AGE



Shower
Longitudinal Development

Often used
but (in my view)
unsatisfactory definition

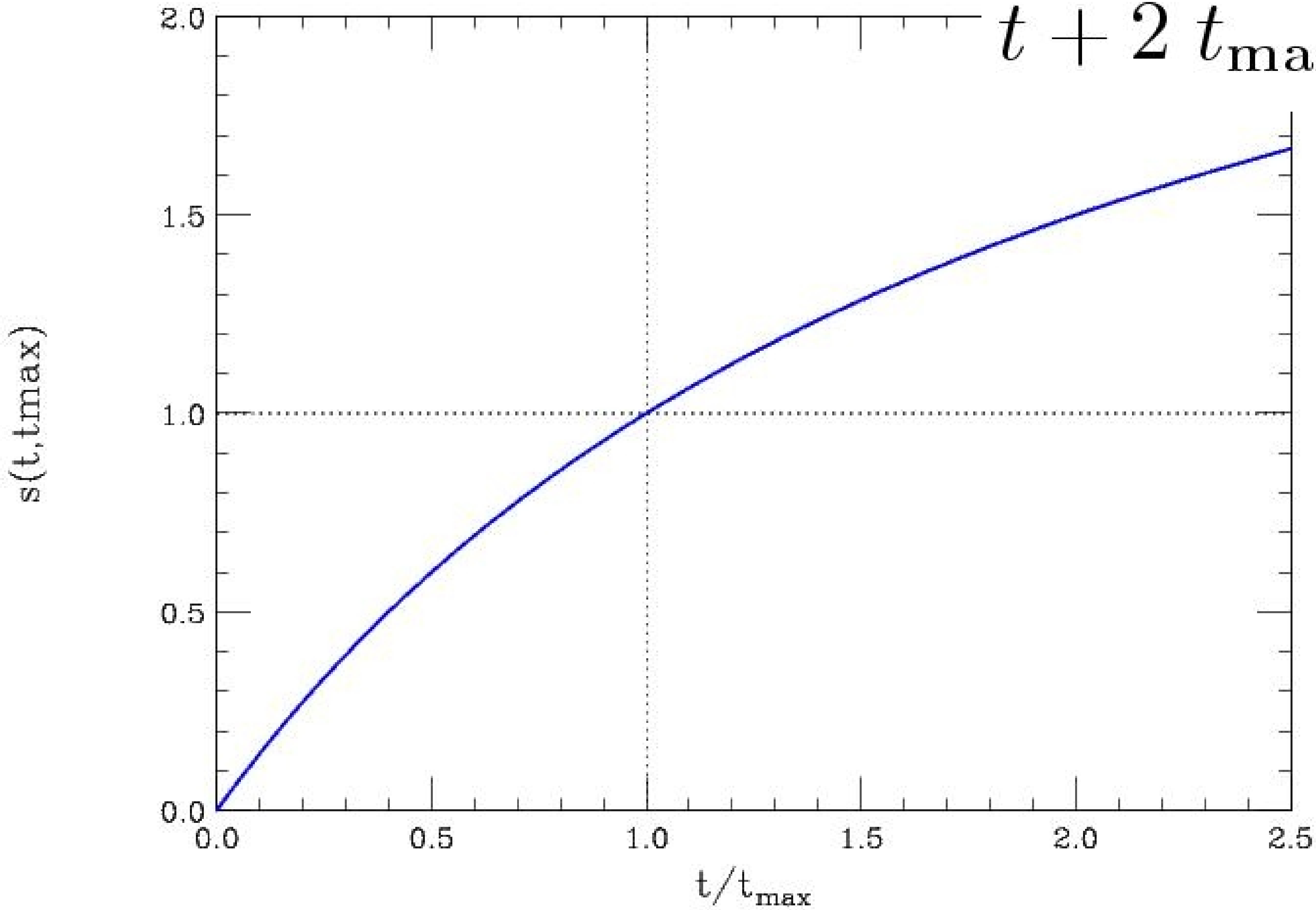
$$S = \frac{3\,t}{t + 2\,t_{\max}}$$

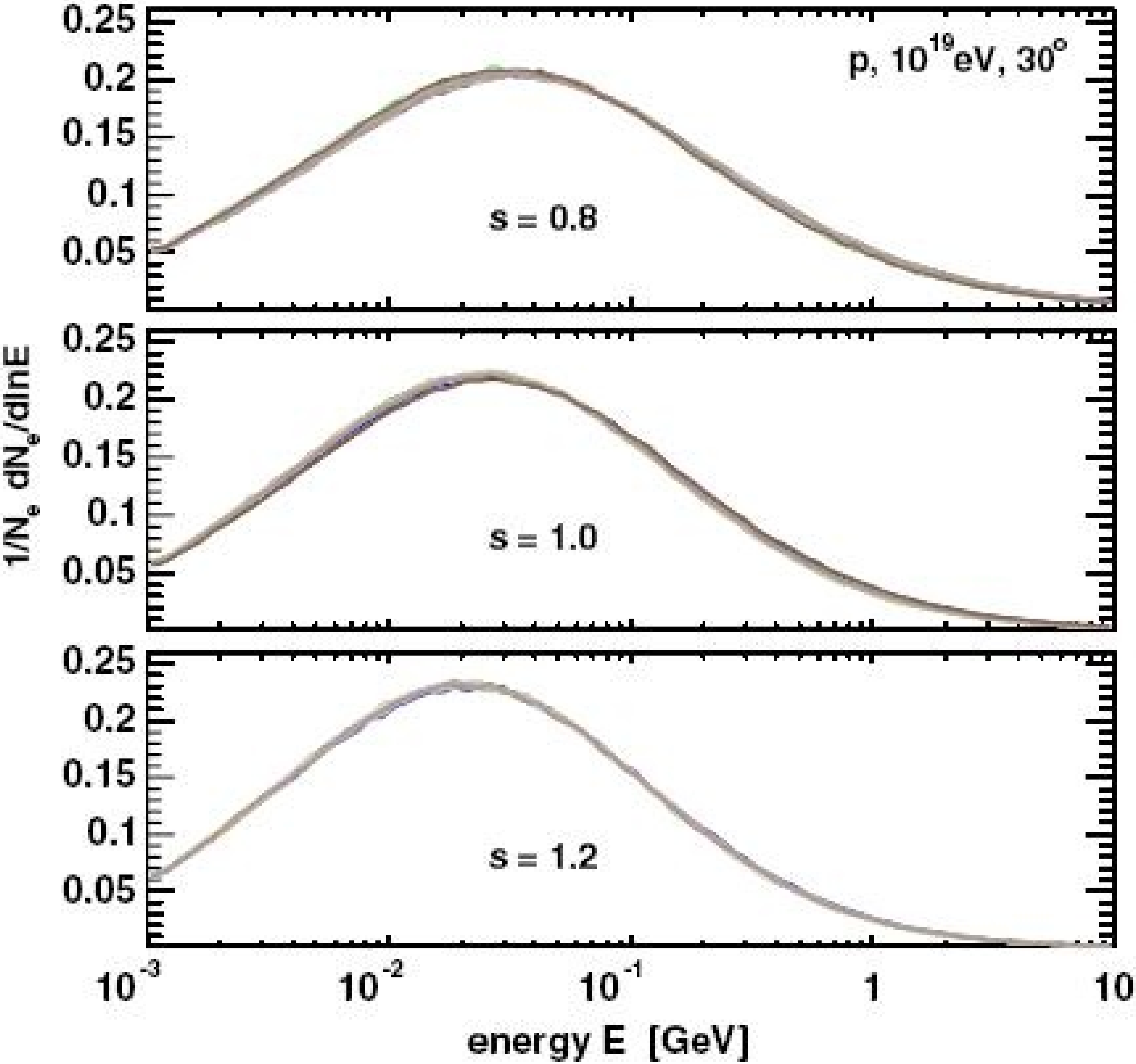
Shower at maximum: $s = 1$

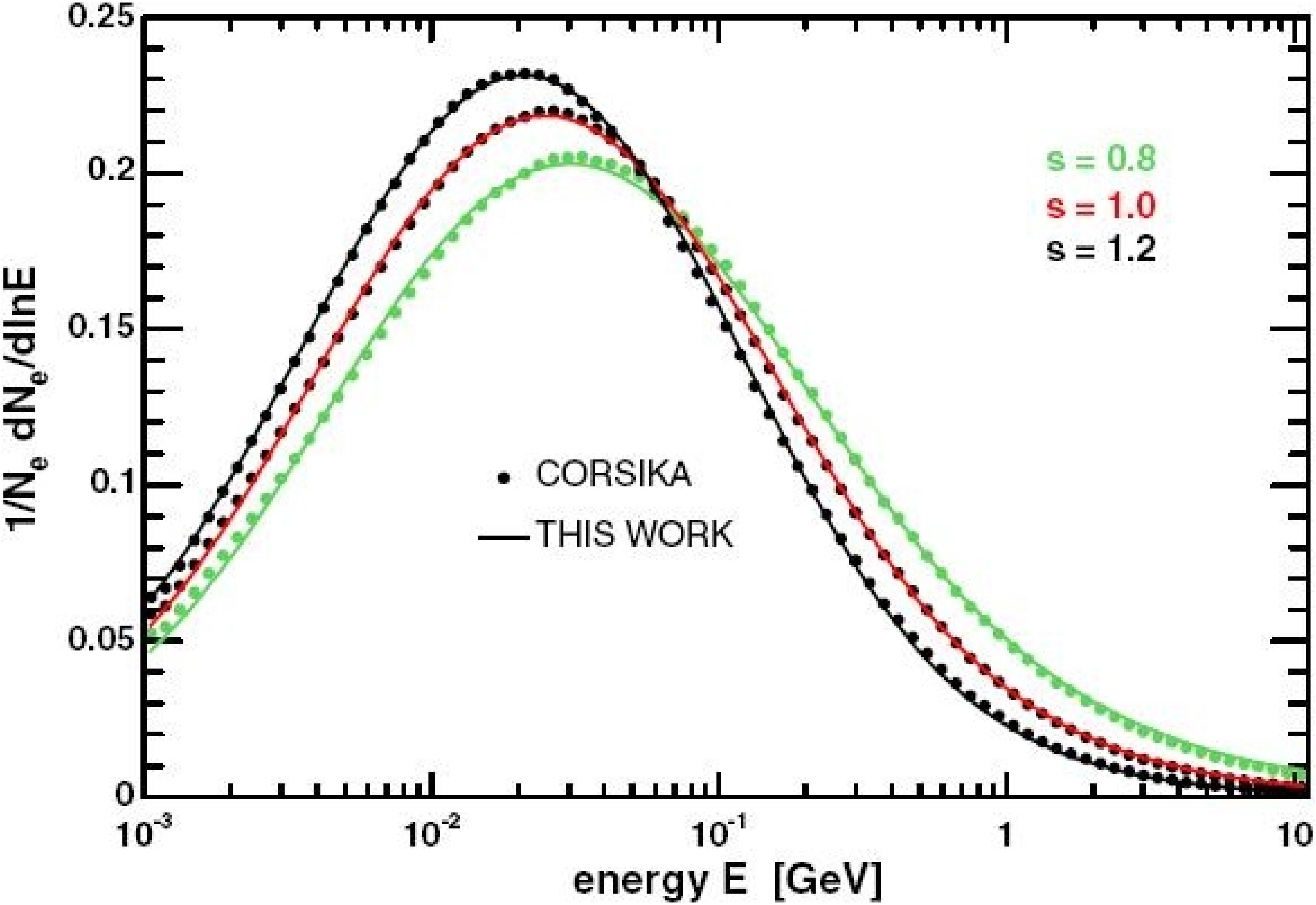
Shower before maximum $s < 1$
Shower after maximum $s > 1$

Age as a function of t/t_{max}

$$\frac{3\,t}{t + 2\,t_{\mathrm{max}}}$$




p, 10^19 eV, 30°
s = 0.8
s = 1.0
s = 1.2
0.25
0.2
0.15
0.1
0.05
1/N_e dN_e/dlnE
10^-3
10^-2
10^-1
1
10
energy E [GeV]



$$f_{\mathrm{e}}(E, s) = a_0 \cdot \frac{E}{(E + a_1)(E + a_2)^s}$$

$$a_1 = 6.42522 - 1.53183 \cdot s$$

$$a_2 = 168.168 - 42.1368 \cdot s$$

with E in MeV

- The shape of the electron energy spectrum is determined (in good approximation) by the "shower Age"
- The Photon spectral shape is (in good Approximation) also determined by the shower Age
- The Ratio photon/Electron is determined by the shower Age

Calculated first by Rossi, Greisen in 1941

"Model Independent " Definition of AGE

$$\lambda = \frac{1}{N(t)} \frac{dN(t)}{dt} \qquad s = \lambda_1^{-1}(\lambda)$$

For real showers the longitudinal development is not identical to the “Greisen Profile” and fluctuates from shower to shower

Violations of the “Universality”

For real showers the longitudinal development is not identical to the "Greisen Profile" and fluctuates from shower to shower

Violations of the "Universality"

$$\lambda = \frac{1}{N(t)} \frac{dN(t)}{dt}$$

$$s = \lambda_1^{-1}(\lambda)$$

General Model Independent Definition of Age

Possible Generalizations:

3-Dimensional treatment.

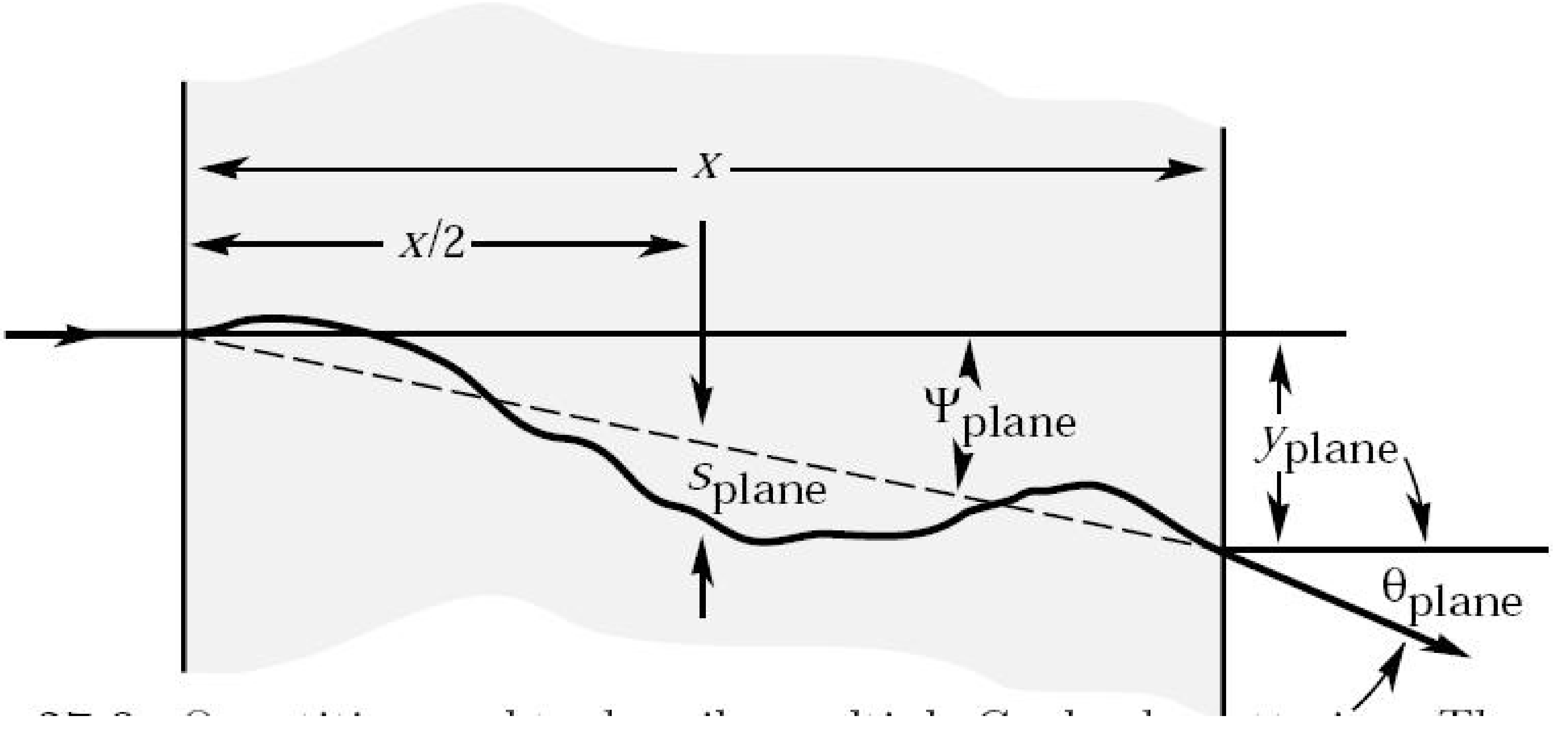
$$n_{e,\gamma}(E, x, \theta_x, y, \theta_y, \ t)$$

Hadronic Showers: add other components

$$n_{p,n}(E,t) \qquad n_{\mu^\pm}(E,t)$$

$$n_{\pi^\pm}(E,t) \qquad n_{\nu}(E,t)$$

Multiple Scattering and LATERAL DISTRIBUTION



$$\langle \theta^2 \rangle_{Av\,(t)} = \tfrac{1}{2} E_s{}^2 t / p^2 \beta^2$$

$$w = 2p\beta / E_s$$

The LANDAU equation

$$\frac{\partial F}{\partial t} = -\theta \frac{\partial F}{\partial y} + \frac{1}{w^2} \frac{\partial^2 F}{\partial \theta^2}.$$

Progress of Theoretical Physics, Vol. 7, No. 2, February 1952

On the Theory of Cascade Showers, I

Jun NISHIMURA
Physics Department of Kobe University
and
Koichi KAMATA
Scientific Research Institute

(Received December 31, 1951)

The diffusion equation of the lateral and angular distribution function were given by Landau,[13] and they are

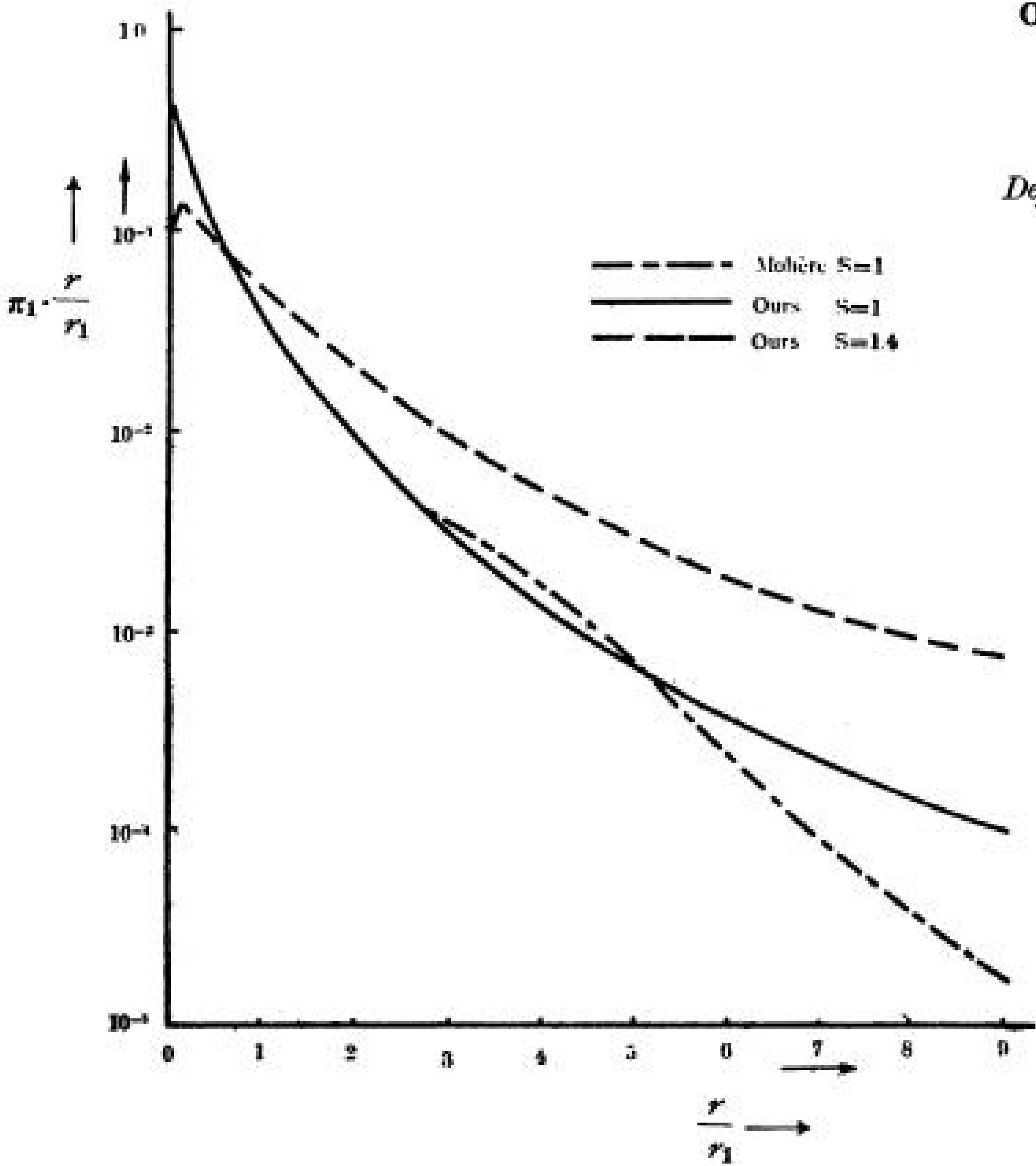
$$\frac{\partial \pi}{\partial t} = -A'\pi + B'\gamma + \frac{K^2}{4E^2}\left(\frac{\partial^2}{\partial \theta_1^2} + \frac{\partial^2}{\partial \theta_2^2}\right)\pi - \left(\theta_1 \frac{\partial}{\partial y_1} - \theta_2 \frac{\partial}{\partial y_2}\right)\pi + \epsilon \frac{\partial \pi}{\partial E} \tag{16}$$

and

$$\frac{\partial \gamma}{\partial t} = C'\pi - \sigma_0 \gamma - \left(\theta_1 \frac{\partial}{\partial y_1} - \theta_2 \frac{\partial}{\partial y_2}\right)\gamma, \tag{17}$$

where

$y_1, y_2, \theta_1, \theta_2$; Lateral and angular deviations of the shower particles from the shower axis.



S = 1.6

S = 1.0

On the Accuracy of the Molière Function, II

J. Nishimura

Department of Physics, Kōbe University

K. Kamata

Scientific Research Institute

July 9, 1951

Nishimura
Kamata

See CONEX lectures of Ralf Ulrich this afternoon.
For applications of these analytic solutions.

[Description of subshowers]
(Alternative to “thinning”.)

Montecarlo tools are extraordinary powerful.
Developing “physical understanding”
Is always very important.