Exact quantum entropy of black holes

2. From gravity to number theory

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Outline

1. Recap of main objectives
2. Macroscopic evaluation of the quantum entropy
3. A universal structure for the gravity path integral
4. New issues – contaminating solutions and wall-crossing
5. Conclusions and future outlook

Outline

Exact quantum entropy

- Macroscopic and microscopic pictures of black holes in string theory. Supersymmetric black holes = collection of zero energy states.
- Microscopic picture gives exact counting formula $d(q, p)$ for the degeneracy of these states as a function of the charges.
- The classical entropy of a black hole agrees with the saddle point approximation for $d(q, p)$ in the large charge limit.
- Perturbative finite size corrections can be understood as inclusion of local higher derivative corrections in the effective action.
- The exact macroscopic quantum entropy $W(q, p)$ is defined as a gravitational functional integral over asymptotically AdS_2 field configurations.

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Main goal

- Compute $W(q,p)$ for arbitrary finite charges by evaluating the functional integral of string field theory on the AdS_2 background.

$$W(q,p) = \Big\langle \exp\big[-i\, q_i \int_0^{2\pi} A^i\, d\theta\big] \Big\rangle_{\mathrm{AdS}_2}^{\mathrm{finite}} .$$

- Compute $d(q,p)$ from bound state dynamics of branes and compare with $W(q,p)$ computed above.

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Main example

Half-BPS states in $\mathcal{N} = 4$ string theory in four dimensions

- These states are specified by two charges (n, w).
- Can be represented as BPS excitations of a fundamental heterotic string on T^6 with winding w and momentum n. The degeneracy depends only on the T-duality invariant $N = nw$. Dabholkar & Harvey [1989]
- String sees 8 transverse spacetime dimensions + 16 internal oscillators = effectively 24 free fields.
- How many ways of distributing energy N in 24 types of oscillators?

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Half-BPS small black hole solution

- The ensemble of half-BPS states above corresponds to a black hole with two charges n and w in four dimensions.
- Classically, this black hole has a singular horizon with zero area and would appear to have zero entropy. However, higher curvature corrections to the supergravity action correct the solution.
- The corrected solution keeping the leading F-type four-derivative term develops a nonsingular horizon with finite string scale area and the geometry has an AdS_2 factor Dabholkar, Kallosh, Maloney [2004].
- The leading Wald entropy of this small black hole is $S_{\mathrm{Wald}} = 4\pi\sqrt{nw}$.
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$N = nw$	$d(N)$	$\log d(N)$	$4\pi\sqrt{N}$	$\log \widetilde{I}_{13}(4\pi\sqrt{N})$
1	24	3.17	12.56	3.94
2	324	5.78	17.77	6.23
3	3200	8.07	21.76	8.31
4	25650	10.15	25.13	10.24
17	6599620022400	29.51	51.81	28.87
18	21651325216200	30.70	53.31	30.03
19	69228721526400	31.86	54.77	31.16
20	216108718571250	33.00	56.19	32.28

Hardy-Ramanujan-Rademacher expansion

Due to its modular symmetries, the degeneracy admits an *exact* expansion

$$d(N) = \sum_{c=1}^{\infty} K\ell(N,-1;c) \left(\frac{2\pi}{c}\right)^{14} \tilde{I}_{13}\left(\frac{4\pi\sqrt{N}}{c}\right)$$

where

$$\tilde{I}_{13}(z) = \frac{1}{2\pi}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{1}{t^{14}}\, e^{t+\frac{z^2}{4t}}\, dt,$$

is a modifield Bessel function of index 13, and

$$K\ell(N,-1;c) = \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^*} \exp\left(\frac{2\pi i d N}{c}\right)\cdot\exp\left(\frac{-2\pi i d^{-1}}{c}\right).$$

is a sum of phases called the "Kloosterman sum".

Outline

Evaluation of the quantum entropy in string theory

- Evaluating the formal expression for $W(q, p)$ by doing the string field theory functional integral is of course highly nontrivial.
- We first integrate out the infinite tower of massive string modes and massive Kaluza-Klein modes to obtain a local Wilsonian effective action for the massless supergravity fields.
- String theory provides a finite, supersymmetric, and consistent cutoff at the string scale. The functional integral with such a finite cut-off and a Wilsonian effective action will be our starting point.
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Even this supergravity functional integral is too hard to solve by presently available techniques. We now proceed in two steps:

Step I: Solve a simpler supergravity problem using localization

We first consider a simpler problem of evaluating the functional integral in a *reduced* theory with $\mathcal{N} = 2$ off-shell supergravity consisting of a gravity multiplet, $n_v + 1$ vector multiplets but no hyper multiplets.

Step II: Lift this back to the full string theory

We then discuss how these results can be used to evaluate the original functional integral in the $\mathcal{N} = 4$ string theory with a specific effective action, including various nonperturbative contributions from orbifolds.

Functional integral in $\mathcal{N} = 2$ off-shell supergravity

- The main advantage of the off-shell formalism is that the supersymmetry transformations are specified once and for all and do not need to be modified as one modifies the action with higher derivative terms.

- Consequently, the localizing solutions that we will describe are universal and do not depend upon the form of the physical action.

- Field content: Weyl multiplet which includes graviton, and $n_v + 1$ vector multiplets

$$\mathbf{X}^I = \left(X^I, \Omega_i^I, A_\mu^I, Y_{ij}^I \right) , \qquad I = 0, \ldots, n_v \, .$$

Here X^I is a complex scalar, A_μ^I a vector field, and Y_{ij}^I are an $SU(2)$ triplet of auxiliary scalars.

Action and black hole solution

- When the action contains only F-type terms, it is completely specified by a single prepotential $F(X^I)$ which is a meromorphic function of its arguments and obeys the homogeneity condition:

$$F(\lambda X^I) = \lambda^2 F(X^I)\,. \qquad (2)$$

- The fields in the near horizon region are

$$ds^2 = v_* \left[-(r^2-1)dt^2 + \frac{dr^2}{r^2-1} \right] + v_* \left[d\psi^2 + \sin^2(\psi) d\phi^2 \right] \;,$$
$$F^I_{rt} = e^I_*, \quad F^I_{\psi\phi} = p^I \sin\psi, \quad X^I = X^I_*, \quad Y^I_{ij} = 0 \;.$$

The values of the constants (e^I_*, X^I_*, v_*) that appear in this solution are determined in terms of the charges (q_I, p^I) by the supersymmetric equations of motion (attractor equations).

Symmetries

- The near-horizon geometry $AdS_2 \times S^2$ has an $SU(1,1|2)$ symmetry.
- This contains the bosonic subgroup $SU(1,1) \times SU(2)$. Conformal symmetry of AdS_2 generated by $\{L, L_\pm\}$, and rotational symmetry of S^2 and is generated by $\{J, J_\pm\}$.
- In addition, there are 8 fermionic symmetries G_r^{ia}, $(i, a, r = 1, 2)$. These obey the commutation relations of the $\mathcal{N} = 4$ superconformal algebra in two dimensions.
- In this theory, we want to compute

$$\widehat{W}(q,p) = \left\langle \exp[-i\, q_i \oint_\theta A^i] \right\rangle_{\mathrm{AdS}_2}^{\mathrm{finite}} . \qquad (3)$$

Localization

Witten[88, 91], Duistermaat & Heckmann [82], Schawarz & Zaboronsky [95]

Consider a supermanifold $\mathcal{M}$ with an integration measure $d\mu$. Let Q be an odd (fermionic) vector field on this manifold satisfying two requirements:

- $Q^2 = H$ for some compact bosonic vector field H,
- The measure is invariant under Q, in other words $div_\mu Q = 0$.

We would like to evaluate an integral of some Q-invariant function $\mathcal{O}$

$$I := \int_{\mathcal{M}} d\mu \, \mathcal{O} \, e^{-\mathcal{S}}.$$

To evaluate this integral using localization, one first deforms the integral to

$$I(\lambda) = \int_{\mathcal{M}} d\mu \, \mathcal{O} \, e^{-\mathcal{S} - \lambda QV} \ ,$$

where $V = (\Psi, Q\Psi)$. Ψ runs over all fermions of the theory, so V is a fermionic, H-invariant function.

- The integral $I(\lambda)$ is independent of λ because

$$\frac{d}{d\lambda}\int_{\mathcal{M}} d\mu\,\mathcal{O}\,e^{-\mathcal{S}-\lambda QV} = \int_{\mathcal{M}} d\mu\,\mathcal{O}\,QV\,e^{-\mathcal{S}-\lambda QV} = 0\;,$$

- This implies that one can perform the integral $I(\lambda)$ for any value of λ and in particular for $\lambda \to \infty$.

- Treating $1/\lambda$ as $\hbar$, one can evalute the functional integral semiclassically. *The semiclassical approximation is exact*.

- The functional integral localizes onto the critical points $\mathcal{M}_Q$ of Q. We shall refer to as these solutions $Q\Psi = 0$ as localizing solutions.

$$I = \int_{\mathcal{M}_Q} d\mu_Q\,\mathcal{O}\,e^{-\mathcal{S}}, \tag{4}$$

with a measure $d\mu_Q$ induced on the submanifold by the original measure.

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Application to supergravity

- In our case, $\mathcal{M}$ is the field space of off-shell supergravity, $\mathcal{S}$ is the off-shell supergravity action with appropriate boundary terms, $\mathcal{O}$ is the supersymmetric Wilson line.
- We need to pick a subalgebra of the full supersymmetry algebra discussed above, whose bosonic generator is compact. The generator

$$Q = G_+^{++} + G_-^{--} \tag{5}$$

squares to $4(L - J)$ which is the generator of a compact bosonic symmetry as desired.

- With this choice for Q, we have to solve for

$$Q\Psi = 0\,. \tag{6}$$

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Off-shell supersymmetry and 1/2-BPS solutions

- We will assume that the metric fields are not excited in the off-shell solution.
- We thus look for BPS solutions purely in the vector multiplet sector, keeping the background AdS_2 geometry:

$$0 = \delta\Omega_i = 2\not{D}X\epsilon_i + \frac{1}{2}\varepsilon_{ij}F_{\mu\nu}\gamma^{\mu\nu}\epsilon^j + Y_{ij}\,\epsilon^j\,,$$

where ϵ_i is the supersymmetry parameter.

- Most general solution:

$$X^I = X^I_* + \frac{C^I}{\cosh(\eta)}\,, \qquad Y^{I1}_1 = -Y^{I2}_2 = \frac{2C^I}{\cosh(\eta)^2}\,, \qquad F^I_{\mu\nu} = F^I_{\mu\nu\,*}\,.$$

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Localizing 1/2-BPS instanton Solution

- The scalar fields X^I in the vector multiplets are no longer fixed at the attractor values X^I_* but have a nontrivial position dependence in the interior of the AdS_2.
- The scalar fields "climb up" the potential. The Q supersymmetry is still maintained because some auxiliary fields also get nontrivial position dependence.
- The real parameters $\{C^I\}$ are the collective coordinates of the localizing instantons. The infinite-dimensional functional integral localizes onto a finite number of ordinary bosonic integrals over $\{C^I\}$.
- To obtain the integrand over this localizing integral, one must substitute this solution into the physical action and extract the finite part as a function of the collective coordinates $\{C^I\}$.

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- The real parameters $\{C^I\}$ are the collective coordinates of the localizing instantons. The infinite-dimensional functional integral localizes onto a finite number of ordinary bosonic integrals over $\{C^I\}$.
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Localizing 1/2-BPS instanton Solution

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The $\mathcal{N} = 2$ supergravity action is:

$$
\begin{aligned}
S_{\rm bulk} &= (-i(X^I \bar{F}_I - F_I \bar{X}^I)) \cdot (-\frac{1}{2}R) + \big[i\nabla_\mu F_I \nabla^\mu \bar{X}^I \\
&+ \frac{1}{4} i F_{IJ} (F^{-I}_{ab} - \frac{1}{4}\bar{X}^I T^{ij}_{ab}\, \varepsilon_{ij})(F^{-abJ} - \frac{1}{4}\bar{X}^J T^{ij}_{ab}\, \varepsilon_{ij}) \\
&- \frac{1}{8} i F_I (F^{+I}_{ab} - \frac{1}{4} X^I T_{abij}\, \varepsilon^{ij})\, T^{ij}_{ab}\, \varepsilon_{ij} - \frac{1}{8} i F_{IJ} Y^I_{ij} Y^{Jij} - \frac{i}{32} F\, (T_{abij}\, \varepsilon^{ij})^2 \\
&+ \frac{1}{2} i F_{\widehat{A}} \widehat{C} - \frac{1}{8} i F_{\widehat{A}\widehat{A}} (\varepsilon^{ik} \varepsilon^{jl} \widehat{B}_{ij} \widehat{B}_{kl} - 2 \widehat{F}^{-}_{ab} \widehat{F}^{-}_{ab}) \\
&+ \frac{1}{2} i \widehat{F}^{-ab} F_{\widehat{A}I} (F^{-I}_{ab} - \frac{1}{4}\bar{X}^I T^{ij}_{ab}\, \varepsilon_{ij}) - \frac{1}{4} i \widehat{B}_{ij} F_{\widehat{A}I} Y^{Iij} + \mathrm{h.c.} \big] \\
&- i(X^I \bar{F}_I - F_I \bar{X}^I) \cdot (\nabla^a V_a - \frac{1}{2} V^a V_a - \frac{1}{4} |M_{ij}|^2 + D^a \Phi^i{}_\alpha D_a \Phi^\alpha{}_i)\, .
\end{aligned}
$$

Renormalized action

- After the renormalization described yesterday, one obtains a remarkably simple form for the finite renormalized action $\mathcal{S}_{ren}$

$$\mathcal{S}_{ren}(\phi, q, p) = -\pi q_I \phi^I + \mathcal{F}(\phi, p)$$

with $\phi^I := e_*^I + 2iC^I$ and $\mathcal{F}$ given by

$$\mathcal{F}(\phi, p) = -2\pi i \left[F\Big(\frac{\phi^I + ip^I}{2}\Big) - \overline{F}\Big(\frac{\phi^I - ip^I}{2}\Big) \right] ,$$

where e_*^I are the attractor values of the electric field.

Evaluation of the Wilson line

- The Wilson line expectation value in supergravity takes the general form

$$\widehat{W}(q,p) = \int_{\mathcal{M}_Q} e^{-\pi\phi^I q_I + \mathcal{F}(\phi,p)}\, Z_{det}\, [d\phi]_\mu$$

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Application of formalism to our main example

Quantum entropy of 1/2-BPS black holes

- The $\mathcal{N} = 4$ string theory in four dimensions has one $\mathcal{N} = 4$ gravity multiplet + 22 vector multiplets.

- The effective action for the F-type terms is governed by the prepotential

$$F(X) = -\frac{1}{2}\frac{X^1}{X^0}\sum_{a,b=2}^{23} C_{ab}X^a X^b - \frac{X^1}{X^0}.$$

  where C_{ab} is the intersection matrix for $K3$.

- The small black hole has $q_A = (n, 0, 0, \cdots, 0)$ and $p^A = (0, w, 0, \cdots, 0)$.

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Quantum entropy of 1/2-BPS black holes

- Then we have:

$$\widehat{W}(q,p) = \int [d\phi^0 d\phi^1 \prod_{a=2}^{23} d\phi^a] \, \mathcal{M}(\phi, w) \, e^{-\pi n \phi^0 - 4\pi \frac{w}{\phi^0} + \frac{\pi}{2} \frac{w}{\phi^0} C_{ab} \phi^a \phi^b} \, .$$

- It is a good guess that the determinant factor in the $\mathcal{N} = 4$ theory is unity.

- Assuming that the measure is the one induced by the supergravity theory, we get (after one contour rotation and some Gaussian integrations):

$$\widehat{W}(q,p) = \int_0^\infty \frac{dS}{S^{14}} \exp\left(\frac{\pi n w}{S} + 4\pi S\right) = \tilde{I}_{13}(4\pi\sqrt{nw}) \, .$$

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Outline

Perturbative and non-perturbative corrections

- We saw that the exact microscopic degeneracy for the 1/2-BPS black hole has an expansion of the form:

$$\Omega(Q) = \sum_{c=1}^{\infty} \exp\left(\frac{2\pi\sqrt{Q^2}}{c} + s_1^{(c)} \log(Q^2) + \frac{1}{\sqrt{Q^2}} s_2^{(c)} \ldots \right) . \qquad (7)$$

- This structure holds much more generally, whenever the black hole descends from a black string wrapped around a circle B. Pioline, S.M. [2009].
- We saw that the functional integral over all matter fluctuations with a fixed background AdS_2 geometry effectively sums up all perturbative corrections.
- In general, there could be other saddle point geometries approaching AdS_2 asymptotically.

Semiclassical interpretation

- We found a family of *smooth* solutions to the semiclassical theory labelled by $c \geq 1$ acting as an orbifold on the original AdS_2 geometry.
  S.M., B. Pioline; Banerjee, Jatkar, Sen.
- They have degeneracy $\exp(2\pi\sqrt{Q^2}/c)$, they are all asymptotically AdS_2, but differ in the interior.
- The AdS_2 is quotiented, and there is an accompanying shift in one of the internal circles so that the whole geometry is smooth.
- These solutions account for the subleading terms in the Rademacher expansion of the small black hole partition function.
- The Kloosterman sum has a natural interpretation from the Wilson lines in the orbifold structure.

Outline

Do the microstates really become a black hole?

- Formulation of microscopic partition functions in flat space at weak coupling involves a representation of a generic charged state as a collection of strings, branes, momentum...
- Assumption – at strong coupling, this configuration gravitates and forms a black hole.
- However, there are other solutions in gravity with same charges which contribute to the total degeneracy (Multi-centered black hole bound states).
- Can one characterize the partition function of single centered black holes ?
- In general, this will break the symmetries of the theory.

Moduli dependence of solutions and wall-crossing

- The zero modes of the scalar fields can take any value at infinity, these are collectively called the *moduli*.
- The single centered black hole solution exist everywhere in moduli space.
- The multi-centered solutions only exist in regions of moduli space bounded by co-dimension one surfaces (walls).
- On crossing these walls, the multi-centered black holes decay, while the single centered black holes are *immortal*.
- This can be used to characterize the single centered black holes.

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Modular symmetries of the partition function

- In string theory, the partition function quite generally has a modular ($SL(2,\mathbb{Z})$) symmetry.
- The black hole can be thought of as momentum excitations of an effective string wound around a circle. This has an associated near-horizon AdS_3 geometry.
- Indeed the full partition function is a modular form, and the Fourier coefficients agree with the black hole degeneracy to good approximation.
- At strong gravitational coupling, not all of the excitations of the string form the black hole. Some of the excitations form multi-centered black hole bound states.

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Symmetry broken and restored

- We would like to remove the contributions from the multi-centered black hole, and keep only the single centered contribution that is constant over moduli space. This destroys the modular symmetry of the function – potentially disastrous!

- However, we show that the partition function for single centered black holes is a *mock modular form*. A. Dabholkar, S.M., D. Zagier,

$$\psi_m(\tau, z) = \psi_m^{BH}(\tau, z) + \psi_m^{\mathrm{multi}}(\tau, z)$$

- ψ_m^{BH} can be completed to a modular partition function $\widehat{\psi}_m^{BH}$ by adding a specific non-holomorphic function. The completion obeys the equation

$$\tau_2^{3/2} \, \frac{\partial \widehat{\psi_m^{BH}}(\tau, \overline{\tau})}{\partial \overline{\tau}} = \sqrt{\frac{m}{8\pi i}} \, \frac{p_{24}(m+1)}{\eta(\tau)^{24}} \sum_{\ell \mod (2m)} \overline{\vartheta_{m,\ell}(\tau)} \, \vartheta_{m,\ell}(\tau, z) \, .$$

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Outline

Summary of Technical Results

Localization of the functional integral

- The full functional integral of string field theory on AdS_2 localizes onto the submanifold $\mathcal{M}_Q$ of critical points of Q, a specific supersymmetry.
- We have obtained an analytic expression for a family of nontrivial complex instantons as *exact* solutions to the off-shell equations of motion. These instanton solutions are completely *universal* and independent of the form of the physical action.

Exact Quantum Entropy

- For the small 1/2-BPS black hole in the $\mathcal{N} = 4$ theory, this reproduced the leading Bessel function of the microscopic theory.
- The exact quantum entropy has the form $W(q,p) = \sum_{c=1}^{\infty} W_c(q,p)$, where the subleading terms come from summing over orbifold solutions of the theory.

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Caveats and open technical problems

- One-loop determinants need to be evaluated. This is straightforward in principle, but could be computationally hard in the general case.
- We have ignored hyper multiplets. The off-shell supersymmetry transformations of the vector multiplets do not change by adding hypers. So our localizing instantons will continue to exist. There could however be additional localizing solutions that excite the hyper multiplet.
- D-terms may contribute. This can be systematically taken into account in this formalism. The solution remains unchanged, the renormalized action may change.

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Lessons learnt

- Finite size effects do have a meaning in gravity. Ensemble is important, can be determined by the classical gravity problem.
- General structure of functional integral of quantum gravity, inclusion and meaning of subleading saddle points.
- Clues about non-renormalization theorems of supergravity.
- Example of exact holography – *Organization of microstates in gravity*.
- Mathematical structures – modular symmetries can be used very effectively in organization of the gravity functional integral. Perhaps they are also important for a deeper reason.

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Future directions and program

- Black holes with less supersymmetry, many interesting directions to follow here. Interesting moduli space dynamics, entropy enigmas, . . .
- Higher dimensional holography and dynamics. Very good approximate notion of locality here. AdS_3 may be a good hunting ground, many puzzles which may yield to these techniques and ideas.
- Deeper relations to mathematics – mock modular nature of black holes.