The Next-to-Simplest Quantum Field Theories

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based on `arXiv:0910.0930` – Shailesh Lal, S.R.

CONTEMPORARY HISTORY

- In the past few years, there has been a revival of interest in S-matrix techniques.
- The idea is that if we look at scattering amplitudes in gauge theories or gravity, they show a lot of interesting structure.
- They have properties that are not at all manifest from the Lagrangian but appear when you look at on-shell amplitudes (Parke, Taylor 1986, Bern et al. 1990s, Witten 2003, Britto, Cachazo et al. 2005, Arkani-Hamed et al. 2008)
- This talk is about these properties and the techniques that have been developed around them.

CONTEMPORARY HISTORY II

- All this is work in progress but over the past few years, the community has succeeded in putting together a coherent set of S-matrix techniques.
- Not only are these techniques teaching us new things about perturbative quantum field theory, they are also useful for calculating amplitudes relevant at the LHC.
- These properties are most useful for gravity, very useful for Yang-Mills and not so useful for the scalar ϕ^4 theory.

$$\text{Usefulness}: \ \text{Gravity} \ > \text{Yang} - \text{Mills} \ > \text{Scalars}$$

Outline

PROGRESS IN SCATTERING AMPLITUDES

- Tree Amplitudes
- Loop Amplitudes

SIMPLE QUANTUM FIELD THEORIES

MOTIVATION

PURE $\mathcal{N} = 1, 2$ THEORIES

- New Recursion Relations
- Structure

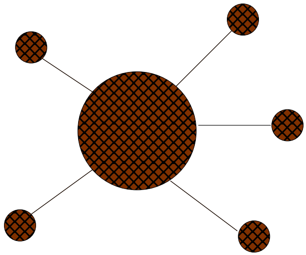
THEORIES WITH MATTER

- The Next-to-Simplest Quantum Field Theories

SUMMARY

SETTING

Figure: SCATTERING AMPLITUDE



- We look at scattering amplitudes of the elementary particles of the theory – gluons for gauge theories, gravitons for gravity etc.
- S-matrix elements are distinct from correlation functions. We get them by putting external legs on-shell
- Scattering amplitudes, but not correlation functions of gravitons and gluons have nice properties.

WHAT NICE PROPERTIES?

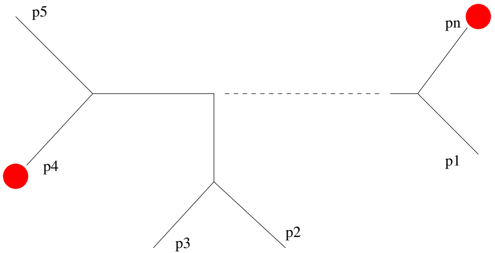
- Consider a tree-level gluon scattering amplitude where 2 gluons have negative helicity have positive helicity?
- One might suspect that this amplitude is a mess. If we have 100,000 gluons, then this amplitude is related to the 100,000 pt correlation function in YM theory. This is ugly even at tree-level!
- Answer is very beautiful and very simple. Called The Parke-Taylor Formula:

$$|M^{--++\cdots}|^2 = \frac{(p_1 \cdot p_2)^4}{(p_1 \cdot p_2)(p_2 \cdot p_3)\ldots(p_n \cdot p_1)}$$

BCFW RELATIONS

- Consider a n-point gluon amplitude.

Figure: BCFW EXTENSION



- Extend *any* two momenta on shell

$$p_4 \to p_4 + qz; \quad p_n \to p_n - qz$$
$$q^2 = q \cdot p_4 = q \cdot p_n = 0$$

- For each p, one of two gauge boson polarization vectors also grows as $O(z)$.

LARGE Z BEHAVIOUR

- How do these amplitudes behave at large z?
- Naive guess:
  - Independent of z for scalars
  - grow fast for gauge theories $O(z^3)$.
  - grow even faster for gravity $O(z^6)$
- Correct Answer: For 3 out of 4 possible polarizations:
  - $M \sim O(1/z)$ for gauge theories
  - $M \sim O(1/z^2)$ for gravity

RECURSION RELATIONS

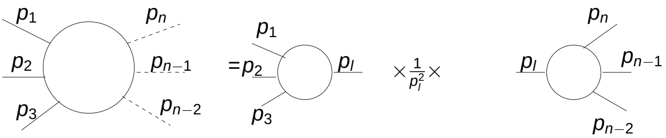
- This property is more than nice. It is very useful.
- The scattering amplitude is a holomorphic function of z. If a holomorphic function dies off at infty, we can reconstruct it from its poles.
- Poles in the amplitude occur when an internal line goes on shell. Residues are lower pt on-shell amplitudes.
- So,

$$M(z) \sim \sum_{\text{partitions}} M_{\text{left}} \frac{1}{P_L^2(z)} M_{\text{right}}$$

- These are the BCFW recursion relations. (Britto et al. '05) They allow us to reconstruct all tree amplitudes from a knowledge of the 3 pt amplitude!

SCHEMATIC BCFW

Figure: Recursion Relations



$+ \ldots$

WHAT USE IS THIS?

- These recursion relations are more than a curiosity.
- They are actually used for calculating scattering amplitudes at the LHC. Far superior to standard Feynman diagram techniques. Large Industry around this (Berger, Bern, Dixon, Forde ...).
- For gravity, these recursion relations are even more useful. Perturbative gravity is a mess!. It has an infinite set of interaction vertices and already there are more than a 1000 terms in the 4-pt interaction.
- Here, everything comes from a 3-pt on-shell function that is determined by Lorentz invariance.

WHY DOES THIS WORK?

- Why do these techniques work and moreover why are they particularly useful for gauge theories and gravity?
- Why are gauge theories and gravity so complicated?
- One reason is that to write a manifestly local description of the theory, we need to introduce redundant degrees of freedom.
- A gluon/graviton has only 2-physical degrees of freedom. To keep locality manifest, we introduce additional degrees of freedom and then try and project them out.

STAY ON-SHELL

- Perhaps, if we can work directly with the physical degrees of freedom, life might be simpler.
- The BCFW recursion relations do this for the classical theory.
- However, they are not manifestly local
- Can these techniques teach you to move away from locality?

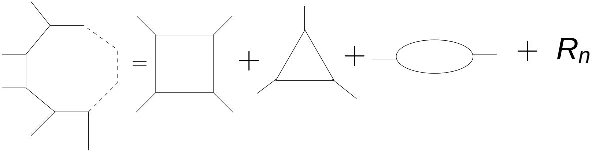
LOOP AMPLITUDES?

- Natural question (precedes grand dreams!): can we generalize on-shell techniques to loop-amplitudes.
- At one-loop, this has been completely worked out.
- At higher loops, we have made some progress but we don't have a complete answer yet.

ON-SHELL TECHNIQUES AT ONE-LOOP

- ANY one loop amplitude in any quantum field theory can be written as a sum of scalar boxes triangles and bubbles with rational coefficients and a possible rational remainder.

Figure: ONE LOOP DECOMPOSITION



- This is surprising. A 1-loop diagram might have 1000 propagators in the denominator. How can we reduce it to something that has at most 4 factors in the denominator

ANALYTIC PROPERTIES OF ONE LOOP AMPLITUDES

- The reason this works is that the analytic structure of the one loop amplitude is tightly constrained.
  - could have branch cut discontinuities: the 2-cut gives us the discontinuity across the branch cut
  - This discontinuity may itself have a discontinuity: cutting three lines gives us the discontinuity of the discontinuity.
  - In 4 dimensions, not more than 4 lines can go on shell.
- So a box plus triangle plus bubble can reproduce the most general branch cut singularities that can appear at one loop.
- A possible rational remainder is accounted for explicitly

LOOPS AND TREES

- To find the S-matrix at one-loop, we need to find the box, triangle and bubble coefficients.
- These coefficients can be found in terms of products of BCFW extended tree amplitudes.
- So, the structure at one-loop is intimately related to the structure of tree-amplitudes under BCFW deformations.

ON-SHELL SUPERSYMMETRY

1, $\frac{1}{2}$

$\frac{-1}{2}$, -1

$\mathcal{N} = 1$ Multiplet

1, $\frac{1}{2}$ $\frac{1}{2}$, 0

0, $\frac{-1}{2}$ $\frac{-1}{2}$, -1

$\mathcal{N} = 2$ Multiplet

1

$4\left(\frac{1}{2}\right)$

$6\,(0)$

$4\left(\frac{-1}{2}\right)$

-1

$\mathcal{N} = 4$ Multiplet

Figure: Representations of SUSY algebra

HOW DOES SUSY HELP?

- In the study of scattering amplitudes, supersymmetry helps in an unusual way.
- Maximal susy implies that every scattering amplitude can be related to a scattering amplitude involving at least two gluons!
- In fact, for maximal susy, we can take both these gluons to be negative helicity gluons
- So, tree amplitudes for $\mathcal{N} = 4$ Super-Yang-Mills and $\mathcal{N} = 8$ Supergravity are the nicest of all! These theories are the Simplest Quantum Field Theories despite having very complicated Lagrangians.

WHAT DOES SIMPLICITY MEAN

Figure: TREE LEVEL

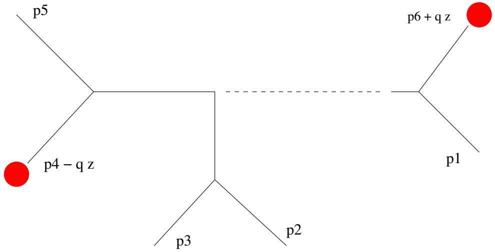
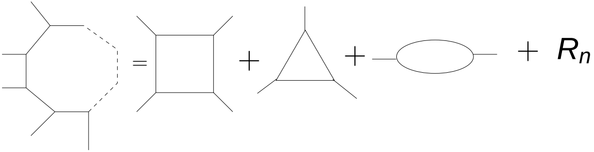


Figure: ONE LOOP



THE MEANING OF SIMPLICITY

- At tree-level, in the $\mathcal{N} = 4$ theory, all amplitudes die off at large z die off under an appropriate BCFW extension.
- The $\mathcal{N} = 4$ theory and $\mathcal{N} = 8$ theory have only boxes at one-loop. Called the no-triangle property.
- This one-loop property is directly linked to the nice tree-level properties of the theory.
- So, the scattering amplitudes of these theories have the simplest analytic structure

THE POINT OF SIMPLICITY

- The fact that $\mathcal{N} = 4$ and $\mathcal{N} = 8$ have such complicated Lagrangians and such simple scattering amplitudes tells us to look for an alternate formulation of these theories.
- Are there any symmetries that can guide us?
- The $\mathcal{N} = 4$ SYM S-matrix has a remarkable symmetry called dual-superconformal invariance which doesn't come from an invariance of the Lagrangian at all!
- Is there any structure at higher loops?
- Many concrete calculations have been done for $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA. This leads to the Leading Singularity Conjecture (Arkani-Hamed et. al '08, '09) for the all-loop S-matrix.

AN EXAMPLE SHOULD BE GENERALIZABLE

- Evidently, $\mathcal{N} = 4$ SYM is the best test-bed to study these S-matrix techniques further. This S-matrix has many wonderful properties, especially when the gauge group becomes large.
- However, it would be disappointing if all this program ended up doing is determining planar gluon amplitudes in maximally supersymmetric SYM.
- We would like to gain some perspective on perturbative quantum field theories. What we learn from these simple theories should be applicable elsewhere.
- This is not a futile hope! The tree-level and one-loop techniques I've described above work for non-supersymmetric gauge theories as well. They even work for noncommutative theories (S.R 2009)

NEXT-TO-SIMPLEST QUANTUM FIELD THEORIES

- So, it makes sense to look for the Next-to-Simplest Quantum Field Theories.
- i.e. theories that are in-between the $\mathcal{N} = 4$ theory and $\mathcal{N} = 0$ theory in terms of complexity.
- It is natural to look at theories with $\mathcal{N} = 1, 2$ supersymmetry.
- It turns out that we can find some next-to-simplest quantum field theories!
- These theories share many of the nice properties of the $\mathcal{N} = 4$ theory. So, they open up new vistas in our study of amplitudes.

ON-SHELL SUSY

- Look again at the structure of these multiplets.

$$\begin{array}{c} 1 \\ \tfrac{1}{2} \end{array} \qquad \begin{array}{c} \tfrac{-1}{2} \\ -1 \end{array}$$

$\mathcal{N} = 1$ Multiplet

$$\begin{array}{ccc} & 1 & \\ \tfrac{1}{2} & & \tfrac{1}{2} \\ & 0 & \end{array} \qquad \begin{array}{ccc} & 0 & \\ \tfrac{-1}{2} & & \tfrac{-1}{2} \\ & -1 & \end{array}$$

$\mathcal{N} = 2$ Multiplet

$$\begin{array}{c} 1 \\ 4\left(\tfrac{1}{2}\right) \\ 6\,(0) \\ 4\left(\tfrac{-1}{2}\right) \\ -1 \end{array}$$

$\mathcal{N} = 4$ Multiplet

Figure: Representations of SUSY algebra

TREE LEVEL: STRUCTURAL SIMILARITY WITH PURE YM

- In $\mathcal{N} = 1, 2$ theories, every scattering amplitude can be related either to one where there are two positive helicity gluons or to one with two negative helicity gluons.
- So, tree-level BCFW relations generalize to $\mathcal{N} = 1, 2$ theories.
- Structurally, these recursion relations are similar to non-supersymmetric YM.
- The two separate multiplets are like the two gluons (of positive and negative helicity) in pure YM.
- Just like pure YM (but unlike $\mathcal{N} = 4$ SYM), not all BCFW extensions lead to good behavior.

ONE LOOP: STRUCTURAL SIMILARITY TO PURE YM

- What happens at one-loop? Do pure $\mathcal{N} = 1, 2$ theories see any simplifications?
- Unfortunately, both triangles and bubbles occur in the one-loop S-matrix of $\mathcal{N} = 1, 2$ theories.
- We should have expected this. The presence of bubbles relates to UV-divergences in the theory.
- We know pure $\mathcal{N} = 1, 2$ theories have UV-divergences, so we should expect bubbles.

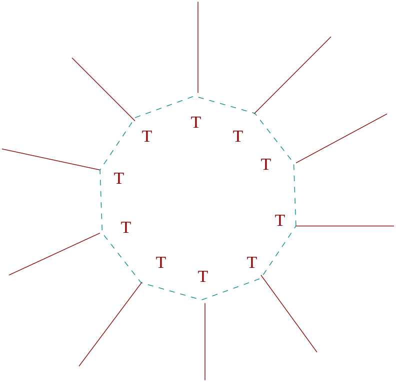
WILL ADDING MATTER HELP?

- This leads to an interesting possibility. What if we look at theories that have a vanishing β function!
- We need to add matter for this.
- Do commonly studied superconformal theories like the Seiberg-Witten theory ($\mathcal{N} = 2$ SU(N) theory with 2 N hypermultiplets) have simple S-matrices?
- These theories do see some simplifications, but not as simple as the $\mathcal{N} = 4$ theory.
- But, by adding different kinds of matter, we can find theories that are even better. Gluon scattering at one-loop is as good as in the $\mathcal{N} = 4$ theory.

Setting

- Consider gauge theories coupled to matter. Lets look at both supersymmetric and non-supersymmetric theories.
- For simplicity, we will focus on gluon amplitudes.
- At tree-level, these are the same in pure YM, YM with matter or $\mathcal{N} = 4$ SYM. So, the question is whether we see structural simplicity at one-loop.

MATTER CONTRIBUTION AT ONE LOOP



$$\sim \mathrm{Tr}\left(T^{10}\right)$$

Figure: Are Matter Contributions Very Complicated?

One Loop Possibilities

- At first sight, the contribution of matter seems very complicated. A 100-gluon amplitude can get a contribution from 100 generators.
- On the other hand, the contribution of matter to the β function is very simple. Matter gives a universal contribution that is proportional to the quadratic index of the matter-representation.
- Turns out that the truth about matter contributions at one-loop is neither very complicated nor very simple. In between the two in a beautiful way!

INDICES: A DETOUR INTO GROUP THEORY

- Recall, that for any representation

$$\mathrm{Tr}_R(T^a T^b) = I_2(R)\kappa^{ab}$$

- Similarly, for $SU(N)$, we can define

$$\frac{1}{2}\mathrm{Tr}_R\left(T^a\{T^b, T^c\}\right) = I_3(R)d^{abc}$$

  I_3 is called the anomaly.

- At higher orders also,

$$\mathrm{Tr}_R\left(T^{(a}T^b T^c T^{d)}\right) = I_4(R)d^{abcd} + I_{2,2}(R)\kappa^{(ab}\kappa^{cd)}$$

- There are as many independent indices as the rank of the algebra.

MATTER CONTRIBUTIONS AT ONE-LOOP: NON-SUPERSYMMETRIC MATTER

For non-supersymmetric theories

- Triangle coefficients depend on the higher order indices up to the sixth order index.
- Bubble coefficients depend on the higher order indices up to the fourth order index.
- The Box-coefficient is sensitive to the entire character.

MATTER CONTRIBUTIONS AT ONE-LOOP: SUPERSYMMETRIC MATTER

For supersymmetric theories

- Triangle coefficients depend on the higher order indices up to the fifth order index.
- Bubble coefficients depend only on the quadratic index.
- The Box-coefficient is sensitive to the entire character.

MIMICKING THE ADJOINT

- The $\mathcal{N} = 4$ theory can also be thought of as a gauge theory with matter in the adjoint representation.
- What if we find a representation whose indices mimic the first few indices of the adjoint?
- Since triangle and bubble coefficients are sensitive only to these indices, such a theory would have a simple S-matrix as well.

CONDITIONS FOR THE S-MATRIX TO SIMPLIFY

Condition (**C**): $\mathrm{Tr}_{\mathrm{R}}(\Pi_{i=1}^{n} T^{a_i}) = m\, \mathrm{Tr}_{\mathrm{adj}}(\Pi_{i=1}^{n} T^{a_i}),\ n \leq p$		
Non-susy theories have	only boxes	no bubbles
if R_f satisfies **C** with	p=6, m=4	p=4,m=4
and R_s satisfies **C** with	p=6, m=6	p=4,m=6.
Susy theories have	only boxes	no bubbles
if R_χ satisfies **C** with	p=5, m=3	p=2,m=3.

Table: Conditions for the S-matrix to simplify

- Since, any representation can be reduced into irreducible representations,

$$R = \bigoplus n_i R_i ,$$

this leads to Linear Diophantine Equations in the n_i

THEORIES WITH ONLY BOXES

1. $\mathcal{N} = 2$, $SU(N)$ (for $N \geq 3$) theory with a symmetric tensor hypermultiplet and an antisymmetric tensor hypermultiplet.
2. More exotic example! Theory based on the gauge-group G_2. Adjoint has dimension **14**. The $\mathcal{N} = 1$ theory, with a chiral multiplet in the representation

$$R_\chi = 3 \cdot [\mathbf{7}] \oplus [\mathbf{27}],$$

3. Commonly studied superconformal theories like the $\mathcal{N} = 2$, $SU(N)$ theory with $2N$ hypermultiplets are not simple in this sense.

THEORIES WITHOUT BUBBLES

- Much easier to find theories without bubbles.
- Any supersymmetric theory with vanishing one-loop β function is free of bubbles at one-loop.
- Can also find non-supersymmetric examples of theories that do not have bubbles.
- Example: Consider $SU(2)$ theory with 7 complex scalar doublets, a pseudo-real scalar in the representation **4** and 4 adjoint fermions.
- Another Example: $SU(N)$ theory with scalar content of a symmetric and anti-symmetric hypermultiplet but 4 adjoint fermions.

OUR SPECIAL $\mathcal{N} = 2$ THEORY

- The $\mathcal{N} = 4$ theory has many nice properties at large N. It has a gravity dual, it shows dual superconformal invariance etc. We hoped that this theory would have these properties too.
- It does! However, this is somewhat trivial. This theory is an orientifold of $\text{AdS}_5 \times S^5$. (Park, Uranga '98, Ennes et al. 2000) Put another way, at large N, it can be obtained from a truncation of the $\mathcal{N} = 4, SU(2N)$ theory.
- Such a theory is called a daughter of $\mathcal{N} = 4$ and inherits the large N properties of $\mathcal{N} = 4$.
- I emphasize that our results go beyond this parent-daughter relationship, since they do not rely on large N.

SUMMARY I

- On-shell techniques hold out the promise of a new formulation of perturbation theory in QFTs
- We have learned many new and surprising things about scattering amplitudes.
- These beautiful structures are most pronounced in the $\mathcal{N} = 4$ SYM theory and $\mathcal{N} = 8$ supergravity. These theories have the simplest scattering amplitudes despite having complicated Lagrangians.

SUMMARY II

- However, these are not exclusive properties of some very special theories.
- Described the generalization of these techniques to theories with less supersymmetry and theories with matter.
- This led us to next-to-simplest Quantum Field Theories. Offer us new test-beds for furthering our study of scattering amplitudes.
- We would like to use all this to understand the reasons for this structure. Holds tremendous promise – sheds new light on QFT, might help us understand gravity and might allow us to move away from locality!