# Neutron superfluidity in the neutron star inner crust

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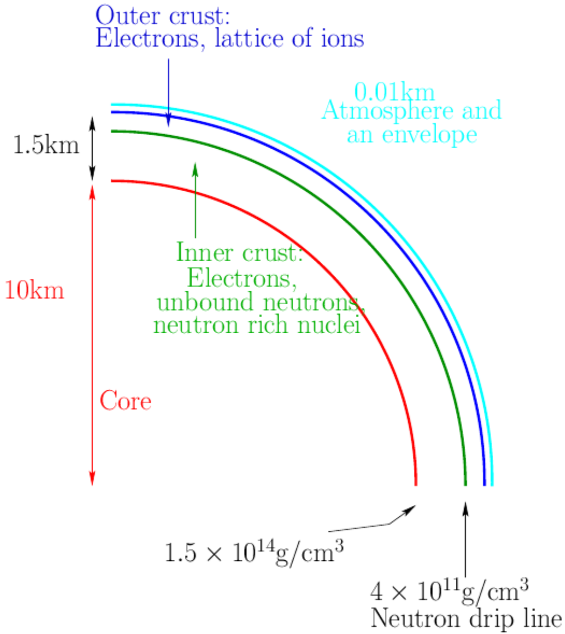
# Neutron star introduction

- A dense stellar remnant containing neutrons, protons, and electrons
- Typical mass $\sim 1.4M_{\odot}$, up to $2 - 2.5M_{\odot}$
- Typical radius $10 - 15$km
- Central density is about $5 - 10\rho_s$ (nuclear density)

# The neutron star inner crust

Unbound neutrons in the inner crust are superfluid. The nuclei form a lattice.



# The neutron star inner crust

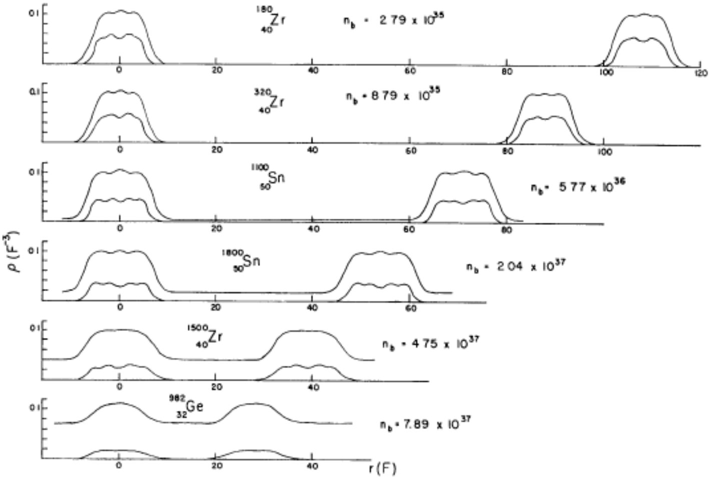



Fig. 3. Proton and neutron density distributions occurring along an axis joining the centers of two adjacent unit cells.

*(Negele Vautherin, PRC (1973))*

# Role of the inner crust

- Cooling of young ($\sim$ 1000 yrs) neutron stars *(Gnedin, Yakovlev, Potekhin, MNRAS (2001))*
- Cooling of transient accreting neutron stars *(Brown, Cumming, APJ (2009))*
- Crustal oscillations in magnetars *(Strohmayer, Watts APJ (2005))*

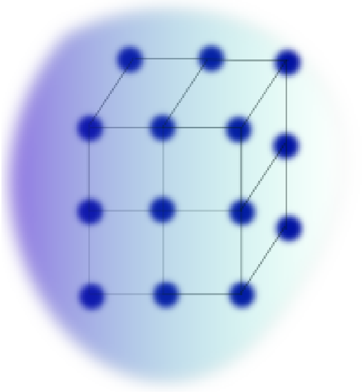
# Summary

- Discuss the low energy theory for the inner crust
  - Relate low energy coefficients (LECs) of the lagrangian to thermodynamic derivatives *(Cirigliano, Reddy, Sharma, Phys. Synopsis. PRC (2011))*
  - Heat transport in the inner crust of magnetars *(Aguilera, Cirigliano, Pons, Reddy, Sharma, Ed. Suggestion. PRL (2009))*
- Response in the absence of the lattice: relation with Unitary Fermi gases
- Dynamics of the Unitary Fermi gas

# The low energy theory for the inner crust

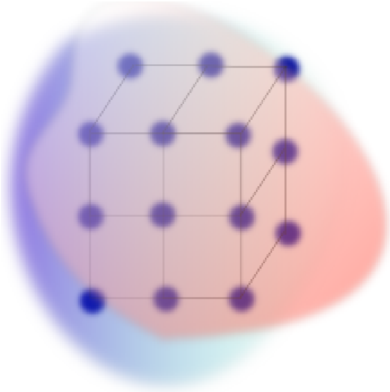
# Ions and electrons

- Nuclei (ions) form a lattice
- Electrons form a nearly free degenerate Fermi gas



# Neutrons

- Neutrons form Cooper pairs
- Breaking a Cooper pair requires energy $\Delta$
- $\Delta \sim 1\text{MeV}$ while $T \sim 0.01 - 0.1\text{MeV}$. Hence neutrons are can not be excited



# Low energy fields

- One Goldstone mode is associated with the phase modulation of the condensate $\langle\psi_1\psi_2\rangle \propto |\Delta| e^{-2i\phi(x)}$
- The second set of Goldstone modes is associated with translations and are the lattice phonons $\xi^a(x)$
- Symmetries require invariance under constant shifts
  - $\phi(x) \to \phi(x) + \theta$
  - $\xi^a(x) \to \xi^a(x) + b^a$

# The effective action

- $$L_{eff} = \frac{f_\phi^2}{2}(\partial_0\phi)^2 - \frac{v_\phi^2 f_\phi^2}{2}(\partial_i\phi)^2 + \frac{\rho}{2}\partial_0\xi^a\partial_0\xi^a - \frac{1}{4}\mu(\xi^{ab}\xi^{ab}) - \frac{K}{2}(\partial_a\xi^a)(\partial_b\xi^b) + g_{\mathrm{mix}} f_\phi\sqrt{\rho}\partial_0\phi\partial_a\xi^a + \cdots$$
- $\xi^{ab} = (\partial_a\xi^b + \partial_b\xi^a - \frac{2}{3}\partial_c\xi^c\delta^{ab})$ is the traceless part of the strain tensor
- An interesting feature is the mixing between the $\phi$ and the longitudinal lattice mode
- The LECs can be related to derivatives of the free energy $\Omega$ with respect to external fields (for eg. the chemical potential $\mu$). We call this thermodynamic matching

# Thermodynamic matching

- Identify the conserved current for the spontaneously broken global symmetry
- Couple an external field to the conserved current, and promote the global symmetry to a local symmetry
- Write a low energy lagrangian for the fields invariant under the local symmetry
- The functional form of the lagrangian to the lowest order is given by the form of the free energy $\Omega$ for constant external fields
- Perform a gradient expansion to relate the low energy constants to thermodynamic derivatives

# Matching for superfluid and crystal

- The external field that couples to neutron number symmetry is $A_\mu$. Path integral for $A_\mu = (\mu, \mathbf{0})$ gives the free energy
- The conserved charge associated with translations is the stress tensor
- The external fields are the spatial components of the external metric $g_{ab}$. $g_{ab} = -[I]$ gives the equilibrium shape of the lattice
- The combination $D_\mu \phi = \partial_\mu + A_\mu$ is invariant under "gauge" transfomations
- The combinations $z^a = x^a - \xi^a(x)$ are invariant under "general covariant" transformations *(Leutwyler Helv. Phys. Acta 1997, Son PRL (2002))*

# Matching for superfluid and crystal

- There are three gauge invariant, scalar combinations
  - $Y = \sqrt{D_\mu \phi D^\mu \phi}$
  - $W^a = \partial_\mu z^a D^\mu \phi$
  - $H^{ab} = \partial_\mu z^a \partial^\mu z^b$

# Matching for superfluid and crystal

- $\mathcal{L}_{\text{eff}}(\phi, \xi^a, A_\mu, g_{\mu\nu}) = f(Y, W^a, H^{ab}) + \dots$
- For constant $g^{ab}(x) = \bar{g}^{ab}$ and $A_\mu(x) = \tilde{A}_\mu = (\mu, \mathbf{A})$ the action at the classical solution at $\phi = 0$, and $\xi^a = 0$ is the free energy
- A new feature is that we need to allow for $\mathbf{A} \neq \mathbf{0}$
- For constant external fields, the variables give $Y_0 = \mu$, $W_0^a = \mathbf{A}^\mathbf{a}$, $H_0^{ab} = \bar{g}^{ab}$
- $f(\tilde{A}_0, \mathbf{A}, \bar{g}_{ab}) = -\Omega(\tilde{A}_\mu, \bar{g}_{ab}) = -\mathcal{E}(\tilde{A}_\mu, \bar{g}_{ab}) + \tilde{A}_\mu j^\mu$

# Quadratic lagrangian

- Expanding near the equilibrium, $Y = \mu$, $W^a = 0$, $H^{ab} = -\delta^{ab}$ and keeping only the quadratic terms

$$
\begin{aligned}
\mathcal{L}_0 = & \frac{1}{2}\Big[\frac{\partial^2 f}{\partial Y^2}\Big](\partial_0\phi)^2 - \frac{1}{2}\Big[\frac{1}{m}\frac{\partial f}{\partial Y} - \frac{\partial^2 f}{3\partial W^c \partial W^c}\Big](\partial_i\phi)^2 \\
& + \frac{1}{2}\Big[\frac{2}{3}\frac{\partial f}{\partial H^{cc}} + m^2\frac{\partial^2 f}{3\partial W^c \partial W^c}\Big]\dot{\xi}^a\dot{\xi}^a \\
& + \Big[\frac{2}{3}\frac{\partial^2 f}{\partial H^{cc}\partial Y} + m\frac{\partial^2 f}{3\partial W^c \partial W^c}\Big](\partial_c\xi^c)(\partial_0\phi) \\
& - \frac{1}{4}[\mu]\,\xi^{ab}\,\xi^{ab} - \frac{1}{2}[K](\partial_c\xi^c)^2
\end{aligned}
$$

# Entrainment

- $n_b = m\frac{\partial^2 f}{3\partial W^c \partial W^c}$ is the density of the neutrons that are entrained on the lattice
- The density of neutrons that participate in transport is $n_c = \frac{1}{3}\langle j^i j^i \rangle (q = 0) = n_{tot} - n_b$ *(Also see Chamel, Pethick, Reddy PTP (2010))*

# The mixing parameter

- $g_{mix}\partial_0\phi\partial_a\xi^a \equiv \frac{1}{f_\phi\sqrt{\rho}}[m\frac{\partial^2 f}{3\partial W^c\partial W^c}\partial_a\phi\partial_0\xi^a + \frac{2}{3}\frac{\partial^2 f}{\partial H^{cc}\partial Y}\partial_0\phi\partial_a\xi^a]$
- First term related to the change in energy associated with relative motion between the superfluid and the lattice. To see that, note that in the non-relativistic limit $W^a \sim m(-\frac{1}{m}\partial_a\phi - \partial_0\xi^a + \frac{1}{m}\partial_i\phi\partial_i\xi^a)$
- The second term related to static interactions. In the case where one conserved species ($p$) forms the lattice and the second species ($n$) is superfluid, $\delta H^{cc} = -2\frac{1}{n_p}\delta n_p$ or $\frac{2}{3}\frac{\partial^2 f}{\partial H^{cc}\partial Y} = n_p\frac{\partial n_n}{\partial n_p}$.

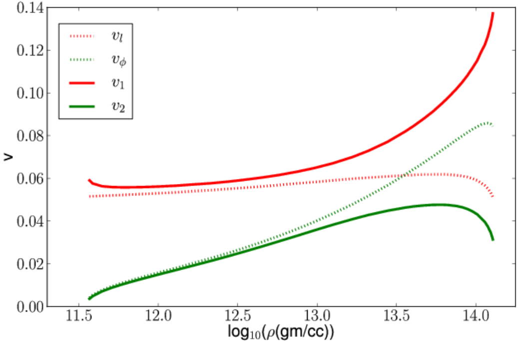
# LECs in the neutron star crust

- $g_{\rm mix} = \frac{1}{f_\phi\sqrt{\rho}}[n_b - n_p\frac{\partial n_n}{\partial n_p}]$
- Use nuclear mass fomulae to obtain a rough estimate for $n_n$ and $n_p$ as a function of density
- We take $n_b$ as the density of bound neutrons ($E_n < 0$) in a single particle, Wigner-Seitz approximation
- The second contribution is estimated by noting that $n_p\frac{\partial n_n}{\partial n_p} \sim n_p f_\phi^2 \tilde{V}_{np}$. For typical values of $\tilde{V}_{np}$, the first term dominates over the second term

# The mixing between modes

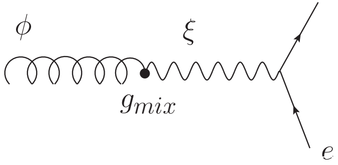
- $$\begin{pmatrix} \xi_k & \varphi_k \end{pmatrix} \begin{bmatrix} \omega^2 - (c_\xi^2)k^2 & (-g_{mix})\omega k \\ (-g_{mix})\omega k & \omega^2 - (c_\phi^2)k^2 \end{bmatrix} \begin{pmatrix} \xi_k \\ \varphi_k \end{pmatrix} \quad (1)$$

# Mixing in the neutron star crust

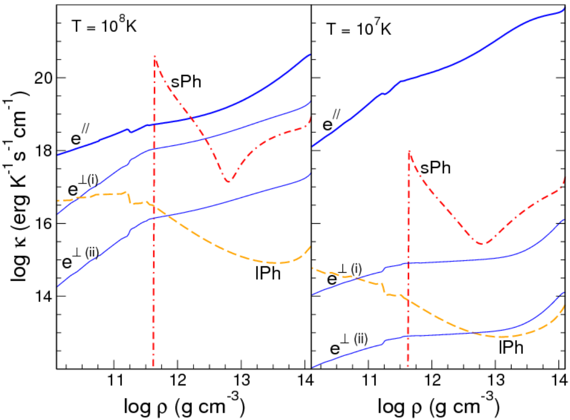


# Effects of mixing

- $C_{v\phi} = \frac{2\pi^2 T^3}{15}(\frac{1}{v_\phi^3})$
- $\lambda_\phi(\omega) = \frac{v_s^2}{g_{\mathrm{mix}}^2}\left[\frac{1+(1-\alpha^2)^2(\omega\tau_\xi)^2}{\alpha(\omega\tau_\xi)^2}\right]\lambda_\xi(\omega)$
- $\lambda_\xi(\omega) \sim \frac{2}{\pi\omega}$
- $\alpha = \frac{v_\xi}{v_\phi}$
- Thermal conductivity $\kappa = \frac{1}{3}C_{v\phi}v_\phi\lambda_\phi$

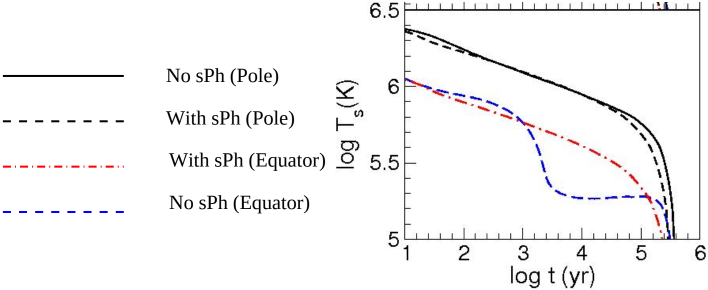


# Conductivity




- $B = 10^{13} - 10^{14}$
- *Aguilera, Cirigliano, Pons, Reddy, Sharma, PRL (2009)*

# Temperature profiles



# Better calculation of $n_b$

- The biggest uncertainty is $n_b$
- A quantum calculation with the full band structure without making a Wigner Seitz approximation is desirable
- One can then calculate the correlation function $n_c = n_n - n_b = \frac{1}{3}\langle j^i j^i \rangle (q = 0)$ *(Chamel et. al.)*
- $n_b$ also affects the frequencies of crustal oscillations

# Dynamics of the unitary Fermi gas

# Relation with the unitary Fermi gas

- Let us consider a simpler system of neutrons without the lattice
- Interactions between neutrons specified by a scattering length $a \sim -18.6\text{fm}$ and effective range $r_e \sim 2.2\text{fm}$
- To be compared to interparticle separation $(n)^{-1/3} = (3\pi^2)^{1/3}/k_F$
- At low densities ($k_F \sim 10^{-1}\text{fm}^{-1}$), $|k_F a| \gtrsim 1$, $k_F r_e \lesssim 1$
- Unitary Fermi gas, $k_F a \to \infty$, $k_F r_e \to 0$, has a conformal symmetry
- $P = c_0 m^{3/2} \mu^{5/2}$ where $c_0$ is related to the Bertsch parameter $\xi$, $c_0 = \frac{2^{5/2}}{15\pi^2 \xi^{3/2}}$. Equivalently $\mathcal{E} = \xi \mathcal{E}_{FG}$
- $\mathcal{E}_{FG} = \frac{3}{5} n \frac{k_F^2}{2m} = \frac{3}{5} \frac{\hbar^2}{2m} (3\pi^2)^{2/3} (n)^{5/3}$

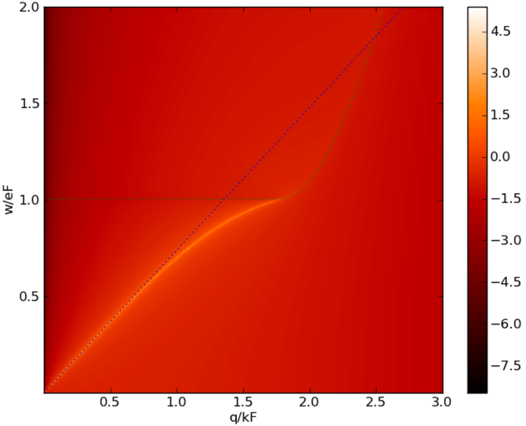
# Effective theory of the unitary Fermi gas

- Constrained due to conformal symmetry
- $\mathcal{L} = c_0 m^{3/2} Y^{5/2} + c_1 m^{1/2} \frac{(\nabla Y)^2}{\sqrt{Y}} + \frac{c_2}{\sqrt{m}} \sqrt{Y} [(\nabla^2 \phi)^2 - 9m \nabla^2 A_0] + \ldots$
- $Y = \mu - V - \dot{\phi} - \frac{(\nabla \phi)^2}{2m}$
- *Son, Wingate Ann. Phys. (2002)*
- Extraction of the constants $c_0$, $c_1$, $c_2$ subtle
- We calculate them using a density functional theory which has been well validated by *ab-initio* calculations

# Superfluid Local Density Approximation (SLDA)

- Kohn-Sham theorem assures that there exists a functional of the density satisfying $d\mathcal{E}/dn = -\mu$
- The form of the functional highly constrained due to conformal symmetry
- $\mathcal{E}[n, \nu] = \alpha \frac{\hbar^2 \tau_r^2}{2m} - \frac{\hbar^2 \gamma}{mn^{1/3}} \nu_r^\dagger \nu_r + \beta \mathcal{E}_{FG}$
- $n = \langle \psi^\dagger \psi \rangle$ and $\nu = \langle \psi \psi \rangle$
- The values of $\alpha = 1$, $\beta = -0.3942$, $\gamma = -13.196$ set to reproduce results of *ab-initio* Monte-Carlo simulations [$\Delta/e_F = 0.502$, $\xi = 0.41$] *(Forbes, Bulgac PRL (2008), Forbes, Gandolfi, Gezerlis PRL (2011))*
- Here we evaluate the linear response $\chi(q, \omega) = \frac{\delta n(q,\omega)}{\delta V(q,\omega)}$

# Response of the unitary Fermi gas

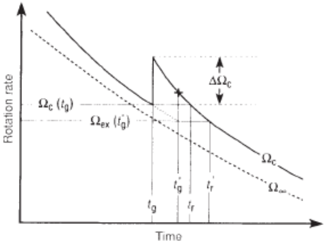



- $c_1 = -0.00498$, $c_2 = 0.00479$
- *Work in progress with M. Forbes (INT)*

# More realistic?

- More complicated SLDA functionals – easy to include
- $\frac{1}{k_F a}$ and $k_F r_e$ corrections – easy to include
- The combined lattice and superfluid problem – hard even at the density functional level, let alone at the *ab-initio* level

# Vortex dynamics in the neutron superfluid




- Neutron star glitches can result from transfer of angular momentum from the neutron superfluid to the rigid crust
- *Link, Epstein, Riper Nature (1992)*

# Multi-vortex dynamics using a unitary bosonic theory

- The SLDA has been successfully used to study the creation of vortices in unitary Fermi gases *(Bulgac et. al. Science (2011))*
- However, glitching involves unpinning of several vortices simultaneously. Interactions between vortices may be important
- On the other hand the short range structure on the vortex may not be very important
- Therefore we try a unitary bosonic model (Bogoliubov-deGennes equations) of time evolution
- $\mathcal{L} = \Psi^* i\partial_t \Psi - \Big[\Psi^*(-\frac{\hbar^2\nabla^2}{4m_F} + 2(V-\mu))\Psi + \xi\frac{3}{5}\frac{\hbar^2}{2m_F}(3\pi^2)^{2/3}(2\Psi^*\Psi)^{5/3}\Big]$
- *(Salasnich, Toigo PRA (2008))*
- One can think of $\Psi$ as the pairing field $\sim \langle\psi\psi\rangle$

# Multi-vortex dynamics using BdG

- Previous studies of multi-vortex dynamics exist *Warszawski, Melatos, Berloff PRB (2012)*
- We are using a BdG equation that respects the symmetry of the problem
- Furthermore, by calibrating our calculations with more microscopic SLDA calculations, we can put in realistic vortex-nucleus interactions
- An example comparison

# Comparing SLDA evolution with GPE evolution

- $v_{\mathrm{stir}} = 0.1 v_F$
- SLDA:Movie
- *(Bulgac et. al. Science (2011))* and *(http://www.phys.washington.edu/groups/qmbnt/UFG/)*
- GPE:Movie
- *(work in progress with M. Forbes)*

# Comparing SLDA evolution with GPE evolution

- $v_{\mathrm{stir}} = 0.2 v_F$
- SLDA:Movie
- GPE:Movie

# Conclusions

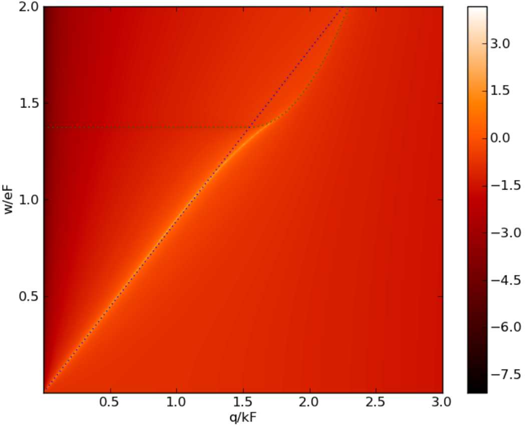
- The existence of neutron superfluidity affects elastic and thermodynamic properties of neutron stars and can affect observables
- Determining the low energy constants can provide a unified treatment of diverse phenomena in the inner crust
- The unitary Fermi gas is a useful model system of superfluid neutrons in the inner crust
- It would be great to definitively show from observations that the inner crusts of neutron stars contain superfluid neutrons
- Shameless advertising: How large a magnetic field needed to break neutron Cooper pairing? See *Gezerlis, Sharma PRC (2011)*

# Backup Slides

# Relating $c$ to response

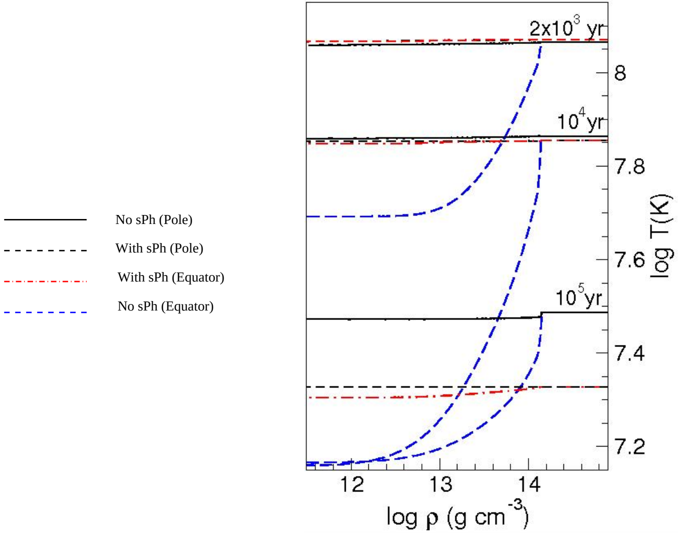
- $\omega(q) = c_s q[1 - \pi^2\sqrt{2\xi}(c_1 + \frac{3}{2}c_2)(\frac{q}{k_F})^2]$
- $c_s = \sqrt{\xi}3\frac{k_F}{m} = \sqrt{\xi}3v_F$
- $\chi(q,0) = -\frac{mk_F}{\pi^2\xi}[1 + 2\pi^2\sqrt{2\xi}(c_1 - \frac{9}{2}c_2)(\frac{q}{k_F})^2]$

# Response of the unitary Fermi gas




- $c_1 = -0.00755547$, $c_2 = 0.0017313$
- *Work in progress with M. Forbes*

# Temperature profiles



# The elastic constants

- $H^{ab} \equiv \eta^{ab} - (\partial^a \xi^b + \partial^b \xi^a) + \partial_\mu \xi^a \partial^\mu \xi^b$ is related to the deformations of the crystal
- The elastic constants are given by,

$$K = \bar{K} + \frac{1}{3} P$$
$$\mu = \bar{\mu} - P$$

where, $P = -\frac{1}{3} \langle T^a_a \rangle$ is the trace of the stress tensor

- $\bar{K} = (\frac{10}{9} \delta_{abcd} - \frac{2}{3} \delta_{ab} \delta_{cd} - \frac{4}{9} \delta_{ac} \delta_{bd}) \frac{\partial^2 \sqrt{-g} f}{\partial g^{ab} \partial g^{cd}}$
- $\bar{\mu} = (\frac{2}{3} \delta_{abcd} - \frac{2}{3} \delta_{ac} \delta_{bd}) \frac{\partial^2 \sqrt{-g} f}{\partial g^{ab} \partial g^{cd}}$

# Formal matching

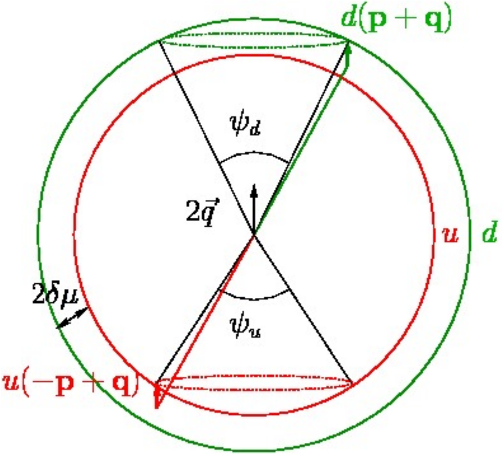
- $Z[A(x)] = \int [d\psi] e^{i \int d^4x \mathcal{L}}$
- For constant $A^\mu(x) = \bar{A}^\mu$, the full integration by definition gives $Z[\bar{A}] = e^{-i\Omega VT}$
- Alternately, do the path integral in two steps. First integrating out the high energy fields, and obtain an effective lagrangian for $\phi$
- Then $Z[A(x)] = \int [d\phi] e^{i\mathcal{L}_{\text{eff}}[\phi, A]}$

# Formal matching, superfluid side

- We can expand the effective action about the classical solution $\phi_0 = 0$. $\phi = \phi_0 + \varphi(x)$
- $e^{if(\mu)VT} \int d[\varphi] e^{i \int d^4x d^4x' \varphi(x)\varphi(x') \frac{\partial^2 \mathcal{L}_{\text{eff}}}{\partial\phi(x)\partial\phi(x')} + \dots}$
- The loop corrections are zero for constant external fields because there is always a derivative acting on the external field
- Therefore $f(Y) = -\Omega(\mu = Y) = P(\mu = Y)$
- *for eg. (Son, Wingate (2006))*

# Introduction to LOFF phases

- BCS pairing is stressed in asymmetric or imbalanced Fermi gases with $(\mu_1 - \mu_2) = 2\delta\mu \neq 0$



# Introduction to LOFF phases

- In electronic systems by applying a magnetic field *(Larkin, Ovchinnikov; Fulde, Ferrell)*
- In dense quark matter by the large strange quark mass *(Rajagopal, Alford, Bowers)*
- In trapped cold Fermi gases by trapping different numbers of $u$ and $d$ *(Pao, Wu, Yip, Mannarelli, Forbes..)*

# Introduction to LOFF phases

- $\Delta(x) = \Delta \sum_{\{\mathbf{q^a}\}} e^{i2\mathbf{q}^a \cdot \mathbf{r}}$
- LOFF phases are possible ground states for $\delta\mu \sim [0.707, 0.754]\Delta_0$, where $\Delta_0$ is the gap in the symmetric phase
- The free energy depends on the set of momentum vectors $\{\mathbf{q^a}\}$ or equivalently the lattice structure
- $|\mathbf{q}^a|$ is chosen to minimize the free energy. $|\mathbf{q}^a| v_f = \eta\delta\mu$ with $\eta \sim 1.2$ and one can compare the free energies of different relative orientations of $\{\mathbf{q^a}\}$
- For simple lattice structures there is a second order phase transition from the normal phase to the LOFF phase at $\delta\mu = 0.754\Delta_0$

# Application to LOFF phases

- We perform the calculation of the low energy constants for a $\cos(2qz)$ condensate
- A Ginzburg-Landau expansion in $\Delta$ can be used near the second order transition

# LECs in the LOFF phase

- $\mathcal{L}_\psi =$

$$\begin{pmatrix} \psi_1^\dagger & \psi_2 \end{pmatrix} \begin{bmatrix} i\partial_t - \frac{p^2}{2m} + (\mu + \delta\mu) & \Delta(x) \\ \Delta^*(x) & i\partial_t + \frac{p^2}{2m} - (\mu - \delta\mu) \end{bmatrix} \begin{pmatrix} \psi_1 \\ \psi_2^\dagger \end{pmatrix}$$

- Evaluate the constants by computing the free energy in the deformed state: $\mathbf{r} \to \mathbf{r} + \xi(r)$
- Expand to the second order in $\xi$ since we want the quadratic lagrangian

$$2\Delta\cos\big(2q(z-\xi)\big) \sim 2\Delta[\cos(2qz) - 2q\xi\sin(2qz) \\ -4q^2\xi^2\cos(2qz)]$$

# LECs in the LOFF phase

- For the cos condensate, one lattice phonon $\xi^z$
- Integrating out the fermions still difficult beccause a space dependent condensate
- Simplify further by making a Ginzburg Landau expansion in $\Delta$, *(Mannarelli, Rajagopal, Sharma (2007))*

# LECs in the LOFF phase

- $\mathcal{L} = \frac{1}{2}[\frac{mk_f}{\pi^2}(\partial_0\phi)^2 - 2\alpha v_f^2(\partial_x\phi)^2 + 2\alpha q^2 v_f^2(\partial_0\xi^x)^2 - 2\alpha q^2 v_f^4(\partial_x\xi^x)^2] + [\alpha q v_f^2(\partial_0\phi\partial_x\xi^x)]$
- $\alpha = \frac{2mk_f\Delta^2}{\pi^2\delta\mu^2(\eta^2-1)}$
- $g_{mix} = \frac{\Delta}{\delta\mu}\frac{v_f}{\sqrt{(\eta^2-1)}}$
- The mixing is parameterically small near the second order point, but may be important when $\Delta$ is larger
- Requires a more careful consideration of gapless fermions