Euclidean Geometry, Analysis and Physics

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Pythagoras

- Natural numbers. Rational numbers p/q. Real numbers.
- Ancients did not know real numbers. Dedekind 1860. Completeness. Set theory.
- Pythagoras (550 BC): $\sqrt{2}$ is irrational. Means, there is no such number!
- Right triangle theorem.
- Crisis!

Euclid

- Euclid Elements (300 BC): Definitions. Axioms. Theorems. Proofs.
- No real numbers, so no distances and lengths.
- Fundamental role of congruence.
- Direct meaning of congruence via ‘motion’.
- Practical measurement of length is indeed via motion!

Group of symmetries

- The set of motions is a group $E(2)$.
- Composite. Associativity, identity, inverse.
- Not commutative.
- In all situations, symmetries form a group.
- In return the properties of group sheds light on the structure.
- Galois 1830.
- Felix Klein 1870: Erlangen program.
- Emmy Noether 1915 : Invariants, conservation laws.

Coordinates

- Descartes, Fermat, 1600.
- How cartesian coordinate frames are laid.
- How the group $E(2)$ looks in coordinate terms.
- Semidirect product $\mathbb{R}^2 \rtimes O(2)$.
- How to compare coordinates.
- Euclidean geometry: study of properties of $\mathbb{R}^2$ and of structures on it, which are invariant under $E(2)$ (look the same in terms of all cartesian coordinate frames).
- Relation with Pythagorean metric $ds^2 = dx^2 + dy^2$.
- The metric gives rise to the affine structure.

Orientation, spin structure

- Left handed, right handed in $\mathbb{R}^n$. Orientation.
- Group $SE(n) = \mathbb{R}^n \rtimes SO(n)$.
- Spin structure on $\mathbb{R}^n$ for $n \geq 2$. Physical meaning. Elie Cartan 1913. Dirac 1928.
- Group $\mathbb{R}^n \rtimes Spin(n)$.
- $Spin(2) = S^1$. $Spin(3) = S^3 = SU(2)$.

Euclidean geometry

- What structures to study on $\mathbb{R}^n$? Subsets. Mountains on a plane. Functions. Effect of $E(n)$.
- The oriented Euclidean line $\mathbb{R}^1$. Time. Sound.
- Pure notes. Effect of translation: just the phase changes. Eigenvectors.
- Fourier 1820. Sound analyzed into pure notes. Fourier transform

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int e^{-i\omega t} f(t) dt$$

- Re-constitution using pure notes: inverse Fourier transform

$$\widehat{f}(t) = \frac{1}{\sqrt{2\pi}} \int e^{i\omega t} \widehat{f}(\omega) d\omega$$

- Decomposition of $L^2(\mathbb{R}^1)$ under translations: Spectral decomposition for id/dx. Momentum in QM.

Decomposition of $L^2(\mathbb{R}^2)$ under $SE(2)$

- Plane waves $e^{ip\cdot x}$. Magnitude $||p||^2 = p_x^2 + p_y^2$ invariant under $SE(2)$.
- Invariant operator of multiplication by $||p||^2$ in dual space ('momentum space').
- Laplacian
$$\Delta = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$
This operator is $SE(2)$-invariant. $L^2(\mathbb{R}^2)$ decomposes as the continuous spectral decomposition of Δ.
- Representation of $\mathbb{R}^2 \rtimes S^1$ on $L^2(S^1)$ by
$$((a,z)\cdot f)(w) = e^{-i\langle p, w^{-1}a\rangle} f(z^{-1}w)$$
- The above for different positive values of $||p||$, together with the characters $\chi_n(a,z) = z^n$ are exactly all the different irreducible unitary representation of $SE(2)$. (Frobenius $\sim$1900. Dirac, Majorana, Wigner: 1928-1937. Mackey machine $\sim$1950: generalization.)

Galilean relativity

- Spacetime $\mathbb{R}^4$. Events as points.
- Inertial observer. Inertial frame. Coordinate transformation. The Galilean group.
- Affine space structure. The constant 1-form dt. Constant Euclidean metric on the kernel of dt. Orientation.
- The Galilean group G as the group of affine transformations of $\mathbb{R}^4$ which preserves orientation, dt and $e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3$. $G = \mathbb{R}^4 \rtimes H \subset \mathbb{R}^4 \rtimes SL(4, \mathbb{R})$.
- Representation theory for G: apply Mackey machine and rep theory for $SO(3)$ and $SE(2)$.

Special relativity

- The Lorentz group came first. Minkowski metric came later!
- $\mathbb{R}^4$ with metric tensor $dt^2 - (1/c^2)(dx^2 + dy^2 + dz^2)$. Tensor $(1/c^2)e_0 \otimes e_0 - e_1 \otimes e_1 - e_2 \otimes e_2 - e_3 \otimes e_3$. Limit as $c \to \infty$ gives the two different tensors of Galileo. Affine structure is a consequence of the metric!
- Lorentz group (proper, orthochronous): $L \subset SL(4, \mathbb{R})$, preserves metric, orientation and time orientation, and fixes a chosen 'origin'.
- Poincare group (proper, orthochronous): $P = \mathbb{R}^4 \rtimes L \subset \mathbb{R}^4 \rtimes SL(4, \mathbb{R})$, preserves metric, orientation and time orientation.
- Sky. The group L as $PSL(2, \mathbb{C})$. Universal cover (spin group) is $SL(2, \mathbb{C})$. $\widehat{P} = \mathbb{R}^4 \rtimes SL(2, \mathbb{C})$ universal cover of P.
- In $2 + 1$ dimension, $L_{2,1}$ is $PSL(2, \mathbb{R})$. Spin group (2-sheeted cover) is $SL(2, \mathbb{R})$.

The main question of physics

- The state of the world, w.r.t. an inertial coordinate frame.
- Question: What will the state be, after some time? Time evolution of the state.
- What is the state w.r.t. other inertial frames.
- Master question: how does the Poincare group P (or rather, $\widehat{P}$) act on the set (or 'space') of all states?
- Quantum mechanics: the state is represented by a non-zero vector (unique up to a scalar multiple) in a Hilbert space. Inner product in the Hilbert space is physically meaningful.
- Unitary representations of $\widehat{P}$. (Projective representations of $SL(2, \mathbb{C})$ necessarily lift to linear reps.)

Wigner's physical interpretation

- Any unitary representation of $\widehat{P}$ decomposes into a 'sum' (direct integral) of irreducible unitary representations.
- Irreducibility must correspond to physical indecomposability. So, these must be the 'elementary' particles.
- Wigner (1940) applied Frobenius theory (Mackey machine) to the semi-direct product $\widehat{P} = \mathbb{R}^4 \rtimes SL(2, \mathbb{C})$ to get its irreducible unitary representations in terms of those of $SU(2) \subset SL(2, \mathbb{C})$ and of $SL(2, \mathbb{R}) \subset SL(2, \mathbb{C})$.
- Bargmann (1947) determined the irreducible unitary representations $SL(2, \mathbb{R})$. Harish Chandra.
- Description of the representations via 'fields' and 'field equations'.
- Mass and spin.
- Spin statistics theorem.

Gauge theories

- Physical properties apart from mass and spin (such as charge) need more structure.
- $\widehat{P}$-equivariant principal bundles on spacetime with other structure groups ('inner degrees of freedom').
- $U(1) \times SU(2) \times SU(3)$. Yang-Mills theories.
- The question of physics remains the same: what is the action of $\widehat{P}$ on the state space.
- Challenge 1: Describe mathematically an interactive quantum field theory on Minkowskian $\mathbb{R}^4$.
- Challenge 2: State the 'main question of physics' in a spacetime which is not Minkowski (as in general relativity), so there is no symmetry group $\widehat{P}$, or even a time translation symmetry.