The pressure of hot QCD

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I. What is it?

QCD:

Quantum field theory describing strong interactions.

$$\mathcal{L}_{\mathsf{QCD}} = \frac{1}{4g^2} \sum_{a=1}^{N_\mathrm{c}^2-1} F^a_{\mu\nu} F^a_{\mu\nu} + \sum_{i=1}^{N_\mathrm{f}} \bar{\psi}_i [\gamma_\mu D_\mu + m_i] \psi_i \ .$$

$N_\mathrm{c} = N_\mathrm{f} = 3$; $m_u, m_d \sim 5$ MeV, $m_s \sim 100$ MeV; $g \gtrsim 1$.

Thermodynamics:

Minus grand canonical free energy density, i.e. pressure.

$$p(T,\mu) \equiv \lim_{V\to\infty} \frac{T}{V} \ln\left\{ \mathrm{Tr}\left[\exp\left(-\frac{\hat{H}_{\mathsf{QCD}} - \mu \hat{B}}{T} \right) \right] \right\},$$

where $\hat{H}_{\mathsf{QCD}}$ is the Hamilton operator corresponding to $\mathcal{L}_{\mathsf{QCD}}$, and $\hat{B}$ is the baryon number operator.

We will denote $p(T) \equiv p(T,0)$. This is the "hot" case. One can also consider the "dense" case $\mu \neq 0$, but in most physics applications $|\mu| \ll T$.

II. Why is it relevant?

In cosmology, the cooling rate of the Universe is

$$\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{\sqrt{24\pi}}{m_{\mathsf{Pl}}}\frac{\sqrt{e(T)}s(T)}{c(T)} \; ,$$

$s = p'(T)$, $e = Ts(T) - p(T)$, $c = e'(T) = Tp''(T)$.

Cosmological relics (dark matter, background radiation, etc) are born when some reaction time $\tau(T)$ becomes longer than the time period $t_{\mathsf{now}} - t(T)$.

For instance, for WIMPs (Weakly Interacting Massive Particles) of mass M, decoupling happens at $T \sim M/25$. For $M = 10 ... 1000$ GeV, $T = 0.4 ... 40$ GeV, in which range QCD effects are important.

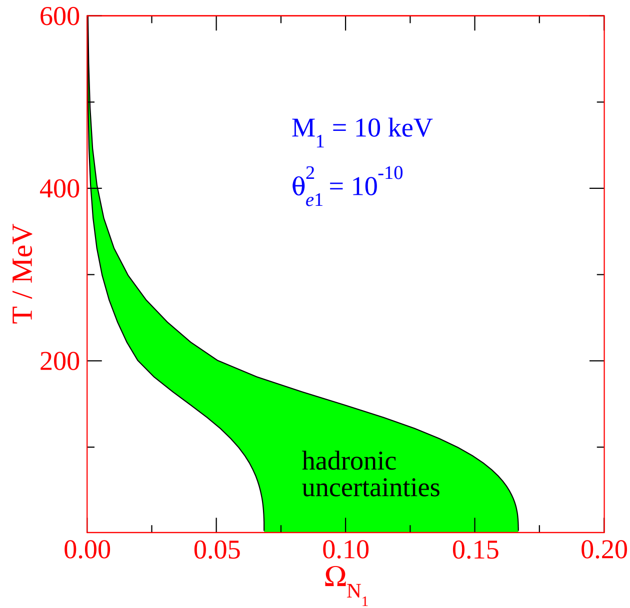
Srednicki Watkins Olive NPB 310 (1988) 693;
Hindmarsh Philipsen hep-ph/0501232

For another candidate, right-handed "sterile" neutrinos with $m_\nu \sim$ keV, decoupling happens at $T \sim 150$ MeV. In this range QCD effects are even more important.

Dodelson Widrow hep-ph/9303287;
Shi Fuller Abazajian astro-ph/9810076, astro-ph/0204293

A concrete example:

Asaka Laine Shaposhnikov hep-ph/0612182



The dark matter relic density is determined to few % by CMB experiments, so theoretical errors should eventually be reduced to the same level.

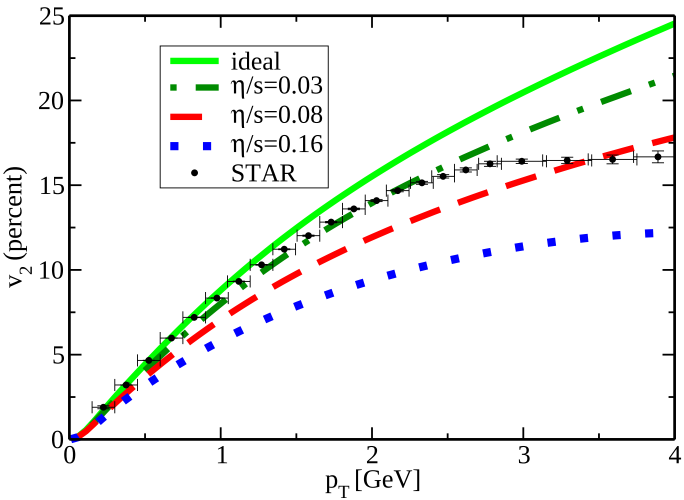
In heavy ion collision experiments, the expansion of the system, after thermalisation, is determined by the energy-momentum tensor

$$T^{\mu\nu} = [p(T) + e(T)]u^\mu u^\nu - p(T)g^{\mu\nu} + \mathcal{O}((\eta, \zeta)\partial^\mu u^\nu) \ ,$$

where u^μ is the flow velocity, and $\partial_\mu T^{\mu\nu} = 0$.

After hydrodynamic expansion the system hadronises at $T \sim 100...150$ MeV. The hadron spectrum observed depends indirectly on $p(T)$.

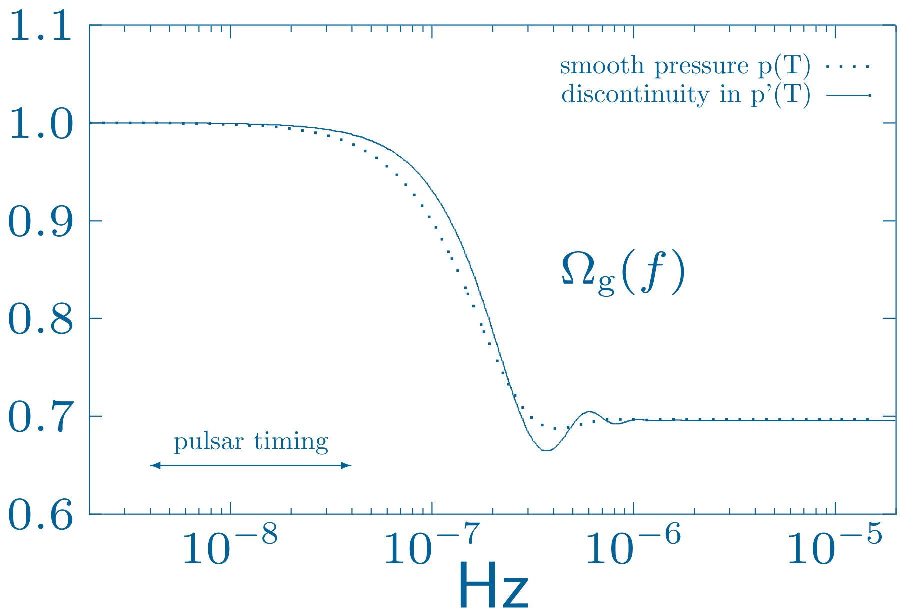
Particularly sensitive is the “elliptic flow”, v_2, characterizing the anisotropy of momentum distribution in various directions:



2×Romatschke 0706.1522

In this example **none** of the models describes the data, so model-$p(T)$ or other inputs need to be modified!

Yet another example from cosmology: Inflation generates a flat spectrum of gravitational waves, but the amplitude decreases once a mode is within the horizon:



Schwarz gr-qc/9709027; Seto et al gr-qc/0305096; Boyle et al astro-ph/0512014

III. Why is it difficult?

Solid understanding is possible in 2 limits only.

0 MeV $\lesssim T \lesssim 100$ MeV: confinement+chiral symmetry breaking $\Rightarrow$ weakly interacting massive hadrons:

$$p(T) \approx \sum_i T^4 \left(\frac{m_i}{2\pi T}\right)^{\frac{3}{2}} e^{-\frac{m_i}{T}} \, .$$

$T \gg 500$ MeV: asymptotic freedom $\Rightarrow$ weakly interacting quarks and gluons:

$$p(T) \approx \frac{\pi^2 T^4}{90}\left[2(N_\mathrm{c}^2 - 1) + \frac{7}{2} N_\mathrm{c} N_\mathrm{f}\right] \approx 5.2 T^4 \, .$$

What happens at intermediate temperatures?

Traditionally it was thought that there is a strong first order phase transition in between the low-temperature and the high-temperature regimes.

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Cosmic separation of phases

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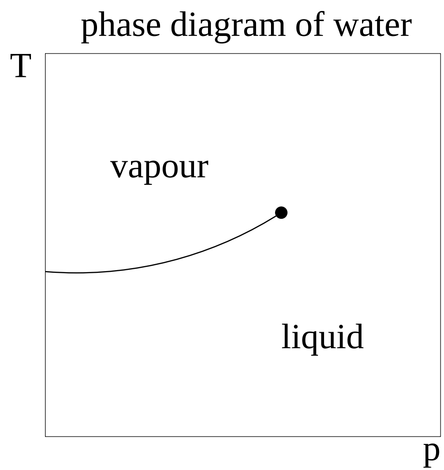
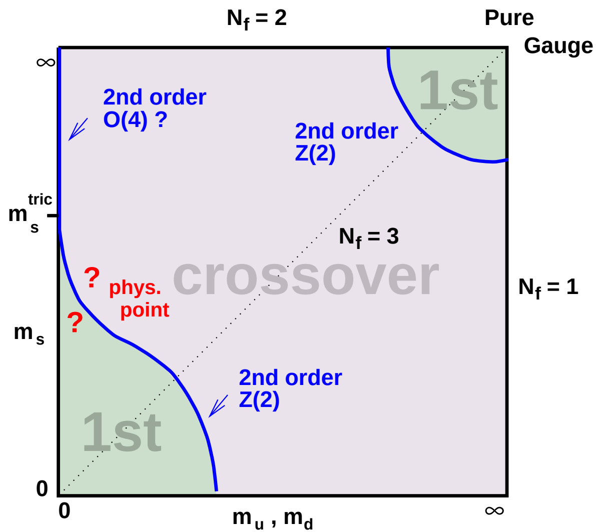
(Received 9 April 1984)

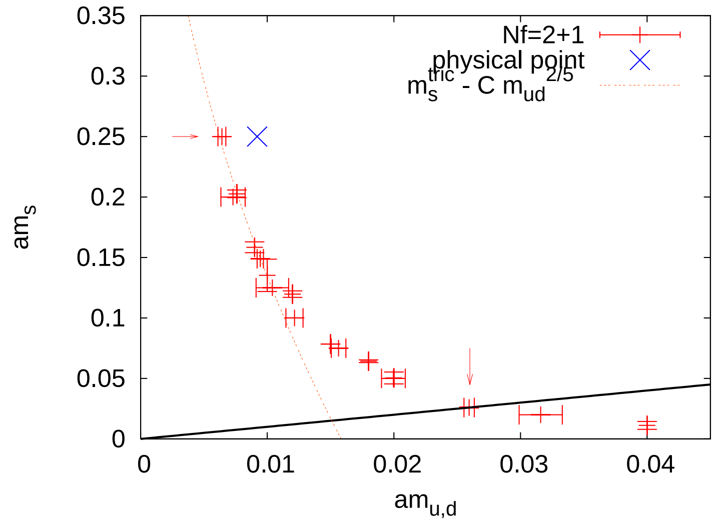
A first-order QCD phase transition that occurred reversibly in the early universe would lead to a surprisingly rich cosmological scenario. Although observable consequences would not necessarily survive, it is at least conceivable that the phase transition would concentrate most of the quark excess in dense, invisible quark nuggets, providing an explanation for the dark matter in terms of QCD effects only. This possibility is viable only if quark matter has energy per baryon less than 938 MeV. Two related issues are considered in appendices: the possibility that neutron stars generate a quark-matter component of cosmic rays, and the possibility that the QCD phase transition may have produced a detectable gravitational signal.

But the distinction between the two limits could also be smooth, like in between liquid and vapour, only with a different label on x-axis: $p \longrightarrow \{m_u, m_d, m_s\}$.



To find out what really happens, lattice experiments have been carried out since more than 25 years.

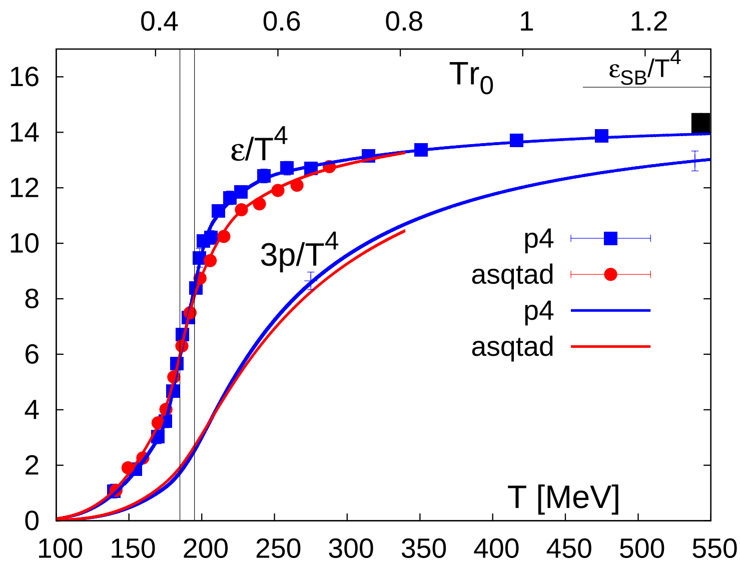




de Forcrand Philipsen hep-lat/0607017; Aoki et al hep-lat/0611014

However systematic errors (volume, lattice spacing, breaking of chiral symmetry) are hard to quantify.

Pressure as a function of T in the crossover region:



hotQCD 0903.4379; Aoki et al 0903.4155; Kanaya et al 0910.5284; Bornyakov et al 0910.2392

Systematic errors possible here as well, but are being reduced through the work of many groups.

Unfortunately numerical results can yield no analytic control over parameteric dependences $(N_c, N_f, m_u, m_d, m_s)$, which would be interesting from the **theoretical** point of view.

N_C-dependence at $N_f = 0$: Datta Gupta 0910.2889; Panero 0907.3719

They are also hard to extrapolate to very high temperatures (1 GeV $\lesssim T \lesssim$ 100 GeV), which would be interesting from the **cosmological** point of view.

So, it may be worthwhile to explore complementary avenues as well!

IV. pQCD at high temperatures

Lattice results deviate from non-interacting quarks and gluons ($\equiv \epsilon_{\mathsf{SB}}(T)$) even at $T \sim 550$ MeV. Could the deviation be understood as a "small correction"?

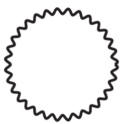
To find out, compute corrections to $p_{\mathsf{SB}}(T)$ in a power series in the QCD coupling constant g!

g^2:	Shuryak 1978; Chin 1978
g^3:	Kapusta 1979
$g^4 \ln(1/g)$:	Toimela 1983
g^4:	Arnold, Zhai 1994
g^5:	Zhai, Kastening 1995; Braaten, Nieto 1995
$g^6 \ln(1/g)$:	Schröder 2002; Kajantie et al 2002
g^6 (partly):	Di Renzo et al 2006; Gynther et al 2009

How does it go in principle?

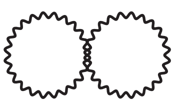
$$p(T) = \lim_{V\to\infty} \frac{T}{V} \ln \int_{\text{b.c.}} \mathcal{D}[A_\mu^a, \bar{\psi}, \psi] e^{-\int_0^{1/T} \mathrm{d}\tau \int \mathrm{d}^3x\, \mathcal{L}_{\text{QCD}}} \; .$$

1-loop:



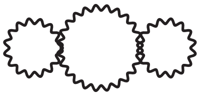
$$p(T) = \frac{\pi^2 T^4}{45} (N_\mathrm{c}^2 - 1 + \frac{7}{4} N_\mathrm{c} N_\mathrm{f}) \; .$$

2-loop:

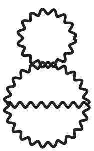


$$\delta p(T) = -\frac{g^2 T^4}{144} (N_\mathrm{c}^2 - 1)(N_\mathrm{c} + \frac{5}{4} N_\mathrm{f}) \; .$$

3-loop: Uncancelled infrared (IR) divergences!



$$g^4 T^5 \int \frac{\mathrm{d}^3 p}{p^4} + \ldots$$ power divergent .



$$g^4 T^4 \int \frac{\mathrm{d}^3 p \, \mathrm{d}^3 q}{p^2 q^2 (p+q)^2} + \ldots$$ log divergent .

Strict perturbation theory breaks down!

Physics: interactions make it a **multiscale** system, generating colour-electric screening at $|\mathbf{k}| \sim m_{\mathsf{E}} \sim gT$, and colour-magnetic screening at $|\mathbf{k}| \sim g^2T/\pi$. We did not account for this, thus met a dead end.

Method for making progress: **effective field theories**.

$$\boxed{\text{QCD; } |\mathbf{k}| \sim \pi T, gT, g^2T/\pi}$$

$\Downarrow$ perturbation theory (1)

$$\boxed{\text{EQCD; } |\mathbf{k}| \sim gT, g^2T/\pi}$$

$\Downarrow$ perturbation theory (2)

$$\boxed{\text{MQCD; } |\mathbf{k}| \sim g^2T/\pi}$$

$\Downarrow$ numerical simulations (3)

$p(T)$.

Contributions: ($\Lambda_\mathsf{E}, \Lambda_\mathsf{M}$ are "matching scales")

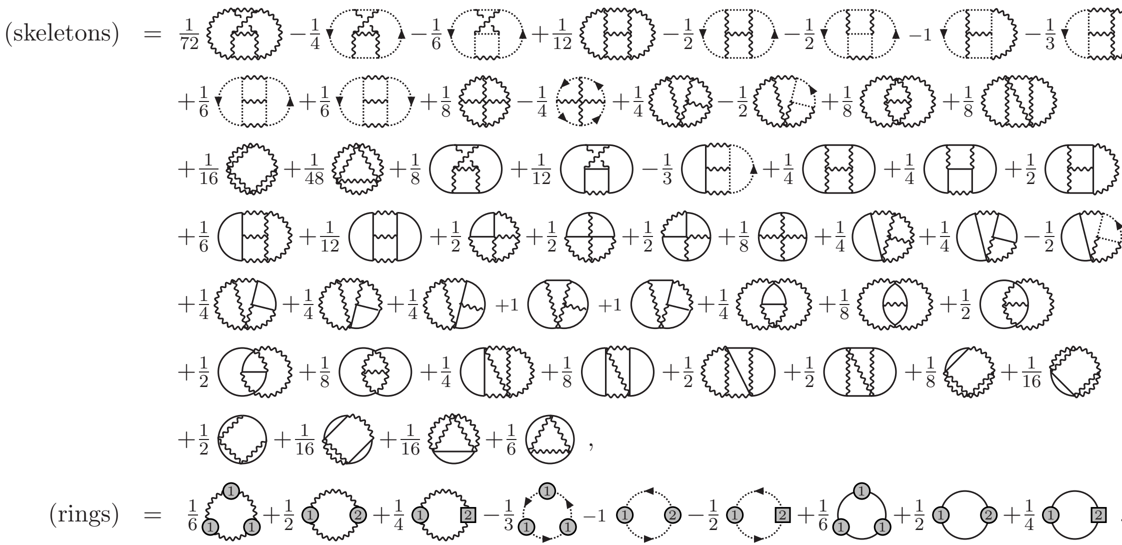
$$\frac{\delta p_{(1)}}{T^4} \sim 1 + g^2 + g^4 \ln\frac{\pi T}{\Lambda_\mathsf{E}} + g^6 \ln\frac{\pi T}{\Lambda_\mathsf{E}} + \ldots\,,$$

$$\frac{\delta p_{(2)}}{T^4} \sim g^3 + g^4 \ln\frac{\Lambda_\mathsf{E}}{gT} + g^5 + g^6 \ln\frac{\Lambda_\mathsf{E}}{gT} + g^6 \ln\frac{gT}{\Lambda_\mathsf{M}} + \ldots\,,$$

$$\frac{\delta p_{(3)}}{T^4} \sim g^6 \left(\ln\frac{\Lambda_\mathsf{M}}{g^2 T/\pi} + [\text{non-pert}] \right)\,.$$

Status: coefficients up to 4-loop logarithms are known analytically, but some "constant" 4-loop terms not.

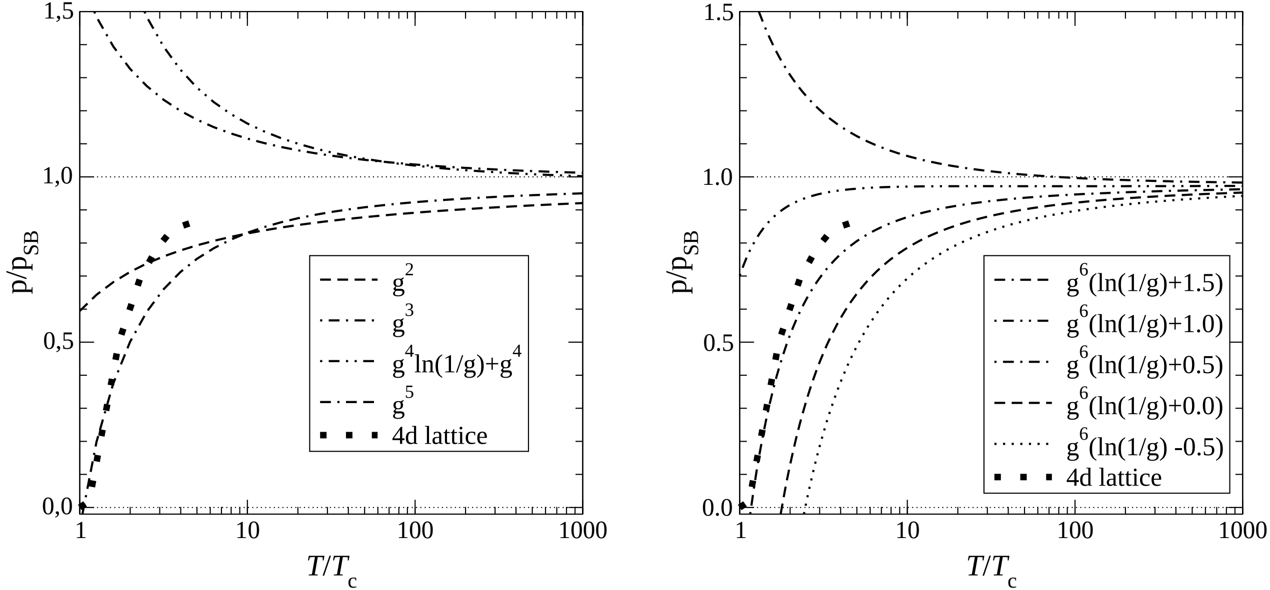
4-loop graphs for the $g^6 \ln(\Lambda_{\mathsf{E}}/gT)$ terms:



Manageable, once automatised routines are employed.

Y. Schröder 2002

Numerical evaluation ($N_{\rm f}=0$):



Kajantie et al hep-ph/0211321

$\Rightarrow$ Interactions are strong even at high temperatures, but at least the g^6-curves have the best shape so far.

Application with $N_\mathrm{f} = 3 - 6$

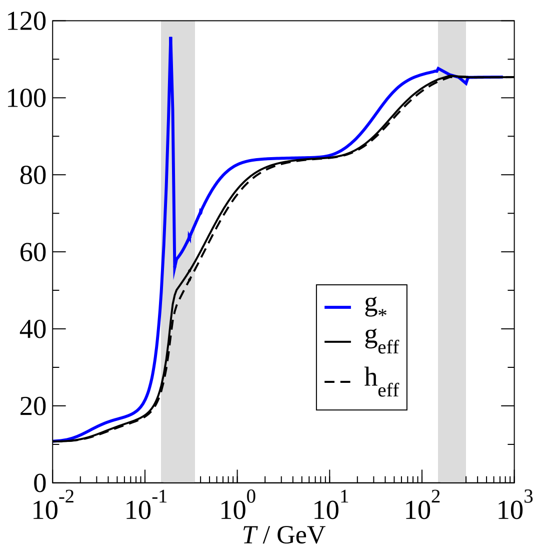
ML Schröder hep-ph/0603048

Evolution equation for $Y \equiv n_\mathrm{dm}/s$: Gondolo Gelmini NPB 360 (1991) 145

$$\frac{\mathrm{d}Y}{\mathrm{d}T} \simeq \sqrt{\frac{\pi g_*(T)}{45}} m_\mathrm{Pl} \langle \sigma v_\mathrm{rel} \rangle (Y^2 - Y_\mathrm{eq}^2) \ ,$$

where $g_* = g_*(p, p', p'')$.

Hindmarsh Philipsen hep-ph/0501232



Physically:
heat capacity has a peak $\Rightarrow$ it takes extra time to dilute all the heat released.

Effects on Y on the percent level?

Conclusions

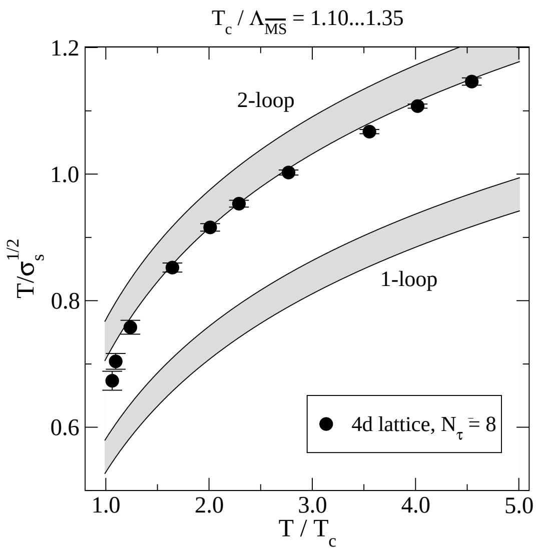
From the theoretical as well as phenomenological point of view, one of the important observables of QCD is minus the free energy density, or pressure, $p(T)$.

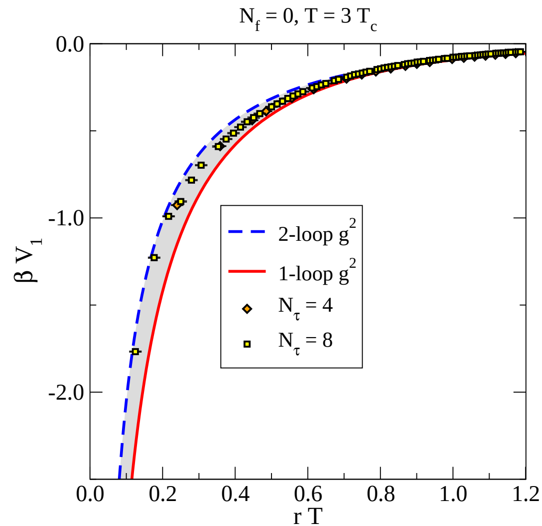
Its computation is however plagued by serious IR sensitivity, even at $T \gg$ GeV.

Effective theory methods allow to cure some of these problems perturbatively, and offer a simple framework for addressing the non-perturbative part numerically.

The results have cosmological significance for certain Dark Matter scenarios.

Successes





Spatial string tension

Heavy quark potential

ML Schröder hep-ph/0503061

Burnier et al 0911.3480