Using Topology to Solve Strongly Coupled Quantum Field Theories

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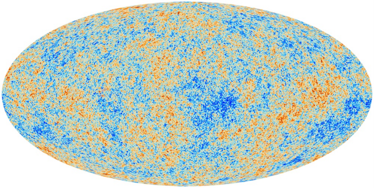
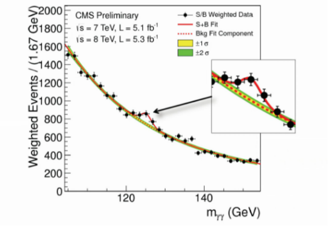
Quantum Field Theory is a universal mathematical structure that follows from two central pillars of modern physics

- Quantum Mechanics
- Special Relativity

It is the main framework in Elementary Particles, Statistical Mechanics, Condensed Matter, Stochastic Processes, and Cosmology.





Many of the BIG open problems in Quantum Field Theory are associated with strong coupling, for example,

- Planck scale physics
- Confinement in Yang Mills theory
- Superconductivity
- Ising model in $d = 3$
- Spin glass...

In this colloquium I would like to describe a technique which allows in some cases to prove exact results about strongly coupled models.

I will start with an easy quantum mechanical model which serves as a pedagogical example and introduces the relevant physical and mathematical machinery.

Incidentally, this model also exhibits the simplest boson-fermion duality I know.

Then I will use an argument from analogy to explain that Yang-Mills theory, which is very important in nature, has the same sort of hidden topology and we can thus extract some non-perturbative results about Yang-Mills theory. I will mention several predictions that follow about the vacuum structure and about the phase diagram.

This is based on several papers in preparation and on a paper that has already appeared in March 2017 with D.Gaiotto, A.Kapustin, and N.Seiberg. The additional collaborators on some other, related, works are J.Gomis, A.Sharon, T.Sulejmanpasic, R.Thorngren, M.Unsal, X.Zhou.

The classical Lagrangian of a particle on a ring is

$$L = \frac{1}{2}\dot{q}^2 - V(q)$$

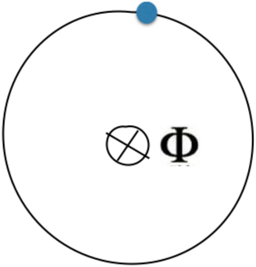
and $V(q)$ being some potential such that $V(q + 2\pi) = V(q)$.

For example for $V = 0$ we could have the particle moving around with arbitrary angular velocity $q = \omega t$, and energy $E = \frac{1}{2}\omega^2$.

The first surprise about this model is that the quantization is ambiguous. There is a one parameter family of choices because the wave function does not need to be periodic

$$\Psi(q + 2\pi) = e^{i\theta}\Psi(q)$$

Physically, this could be due to a solenoid with $\Phi = \theta$ magnetic flux


$\otimes \Phi$

Equivalently, in terms of Feynman's sum over histories, we can break up the space of possible histories according to how many times one winds around the circle and add an appropriate phase. So the Lagrangian is really

$$L = \frac{1}{2}\dot{q}^2 - V(q) + \frac{\theta}{2\pi}\dot{q}$$

and now the wave functions are taken to be always periodic.

The extra term in the path integral,

$$Exp\left[i\frac{\theta}{2\pi}\int dt\dot{q}\right]$$

is just

$$Exp\left[i\theta n\right]$$

where n is the number of time the particular path winds around the circle.

Note that the deformation parameter θ is periodic. This is manifest in all the descriptions we have given so far

$$\theta \simeq \theta + 2\pi \ .$$

Let us take $V = 0$.
The conjugate momentum is shifted as $\Pi_q = \dot{q} + \frac{\theta}{2\pi}$ and the Hamiltonian is thus

$$H = \frac{1}{2}\left(\Pi_q - \frac{\theta}{2\pi}\right)^2$$

The eigenfunctions and eigenvalues are

$$|n\rangle \ , \qquad \langle q|n\rangle = \Psi_n(q) = e^{iqn} \ .$$

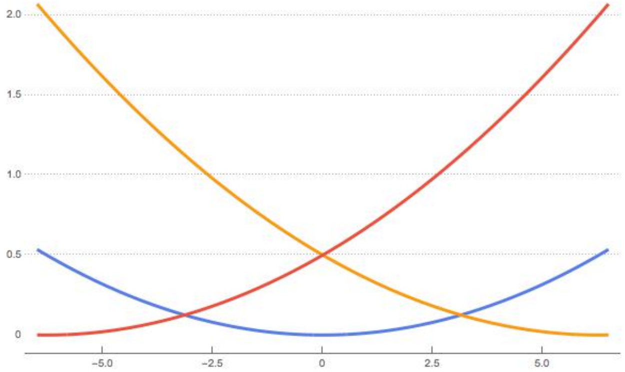
$$E_n = \frac{1}{2}\left(n - \frac{\theta}{2\pi}\right)^2 \ .$$

We see that the full spectrum is indeed invariant under $\theta \to \theta + 2\pi$. But the individual states cross each other so we have a nontrivial map where

$$\theta \to \theta + 2\pi$$

is accompanied by

$$|n\rangle \to |n+1\rangle \ .$$



2.0
1.5
1.0
0.5
0
−5.0
−2.5
0
2.5
5.0

One should say that the theories with θ and $\theta + 2\pi$ are equivalent because there is a similarity transformation between them. The similarity transformation is however nontrivial. Indeed, it is implemented by the operator

$$U = e^{iq} \ , \qquad U^\dagger U = 1 \ .$$

A curious case is $\theta = \pi$ where we see that the ground state is doubly degenerate – it is a qubit. For example at $\theta = \pi$ the ground states are spanned by

$$|0\rangle, |1\rangle \ .$$

So far this could have been an easy exercise in a graduate course. But now let us discuss what happens if we add a potential

$$V(q) = \sum_{k \in \mathbb{Z}} c_k \cos(kq) + d_k \sin(kq)$$

I will introduce some tools that would allow to prove the following theorem:

If only $c_{2k} \neq 0$ then the qubit at $\theta = \pi$ remains.

So for example we expect that $V(q) = cos(q)$ would lift the qubit but $V(q) = cos(2q)$ would retain it.

We begin by considering again the free model $V = 0$. Obviously it has a rotation symmetry $q \to q + \alpha$ for any α. This symmetry group is

$$SO(2)$$

At $\theta = 0, \pi$ there is another symmetry $q \to -q$.

Note that to see that $q \to -q$ is a symmetry at $\theta = \pi$ we used the fact that $\theta = \pi$, and $\theta = -\pi$ describe the same theory. So the symmetry group at $\theta = 0, \pi$ is

$$O(2)$$

The generator of the reflection is denoted by C and the generator of $SO(2)$ is V_α. The group $O(2)$ means that we have

$$CV_\alpha C = V_{-\alpha} \;, \qquad C^2 = 1 \;, \qquad \alpha \simeq \alpha + 2\pi \;.$$

Now we should realize them on the Hilbert space:

$\underline{\theta = 0 :}$ $V_\alpha|n\rangle = e^{in\alpha}|n\rangle,\ C|n\rangle = |-n\rangle.$

$\underline{\theta = \pi :}$ $V_\alpha|n\rangle = e^{in\alpha}|n\rangle,\ C|n\rangle = |-n+1\rangle.$

$\underline{\theta = 0 :}$ $V_\alpha|n\rangle = e^{in\alpha}|n\rangle$, $C|n\rangle = |-n\rangle$.
This is completely consistent with $C^2 = 1$, $\alpha \simeq \alpha + 2\pi$ and $CV_\alpha C = V_{-\alpha}$. The ground state $|0\rangle$ is unique, and $O(2)$ invariant.

$\underline{\theta = \pi :}$ $V_\alpha|n\rangle = e^{in\alpha}|n\rangle$, $C|n\rangle = |-n+1\rangle$.
$C^2 = 1$, $\alpha \simeq \alpha + 2\pi$ are obeyed. But if we try to check the algebra of charges we find that

$$CV_\alpha C = e^{i\alpha} V_{-\alpha}$$

We see that the algebra of operators is centrally extended when we act on the Hilbert space! If we acted on the projective Hilbert space, the ground state qubit would be a perfectly good representation of $O(2)$. But the actual Hilbert space is only a representation of a centrally extended group.

Can we remove this central extension?

$$CV_\alpha C = e^{i\alpha} V_{-\alpha}$$

define $U_\alpha = V_\alpha e^{-i\alpha/2}$, then

$$CU_\alpha C = U_{-\alpha} \ .$$

But now $\alpha \ \simeq \alpha + 4\pi$. So now we see that the central extension leads to a double cover of the group. This is very similar to how $SU(2)$ is the double cover of $SO(3)$. Here we have also a central extension by $\mathbb{Z}_2$

$$\mathbb{Z}_2 \to Pin(2) \to O(2)$$

A neat way to summarize this curious situation is to say that the classical symmetry $O(2)$ suffers from a 't Hooft anomaly in the quantum theory. In particular, the ground state is not a representation of $O(2)$ but of $Pin(2)$. It is therefore natural to give the $|0\rangle, |1\rangle$ states charge $1/2$ under the rotations. So the particle $q(t)$ is really a **fermion** at $\theta = \pi$.

In fact, the ground state of the model is dual to the free fermion

$$\mathcal{L} = \psi^\dagger \dot{\psi}$$

The naive symmetries of a qubit are charge conjugation $\psi \to \psi^\dagger$ and phase rotations $\psi \to e^{i\alpha}\psi$ but in the quantum theory there is a central extension, exactly like for the symmetries of the boson!

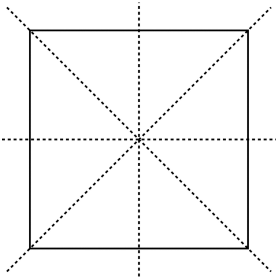
We are thus dealing here with a system that has a $\mathbb{Z}_2$ anomaly for its $O(2)$ global symmetry. But do we really need to preserve the full $O(2)$ for that? Suppose we only retain V_π and C. Classically they just generate $\mathbb{Z}_2 \times \mathbb{Z}_2$. Quantum mechanically we have a central extension η.

$$V_\pi^2 = C^2 = 1 \ , \qquad CV_\pi C = \eta V_\pi \ , \qquad \eta^2 = 1 \ ,$$

and $\eta = -1$ at $\theta = \pi$ and $\eta = 1$ for $\theta = 0$. η is central.

The group generated by V_π, C, η is just the dihedral group that preserves the square, D_8.
So while the symmetry at $\theta = 0$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$, at $\theta = \pi$ it is centrally extended to D_8.

The ground state qubit $|0\rangle \oplus |1\rangle$ furnishes the two dimensional representation of D_8 where $|0\rangle$ and $|1\rangle$ are like the sides of a square.



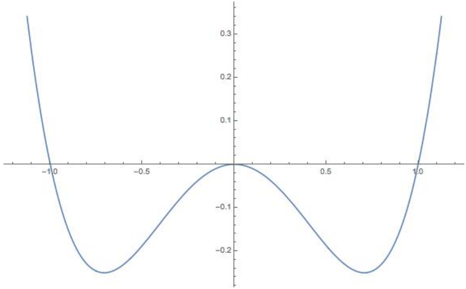
Since our $\mathbb{Z}_2$ anomaly exists even if we break $O(2)$ down to $\mathbb{Z}_2 \times \mathbb{Z}_2$, we can now add any potential which has this symmetry. Since the anomaly cannot disappear the degeneracy has to remain! For example, any potential of the form

$$V(q) = \sum_{k \in \mathbb{Z}} c_{2k} \cos(2kq)$$

would work since it preserves $q \to -q$ and $q \to q + \pi$.

Note that we have an example here, of a potential with two minima, e.g. $cos(2q)$, where the degeneracy between the two classical minimal is preserved in the exact quantum theory. The folklore is that instantons always lift the degeneracy in quantum mechanics and the ground state is unique (at least in non-supersymmetric cases). We see that this is not true.

Why is it true then in potentials like the double well?



Here indeed the ground state is $\Psi_- + \Psi_+$, it is $\mathbb{Z}_2$ invariant, and it is non-degenerate.

It is because $\mathbb{Z}_2$ cannot have nontrivial extensions, as simple as that...

$$H^2(\mathbb{Z}_2, U(1)) = 0 \ .$$

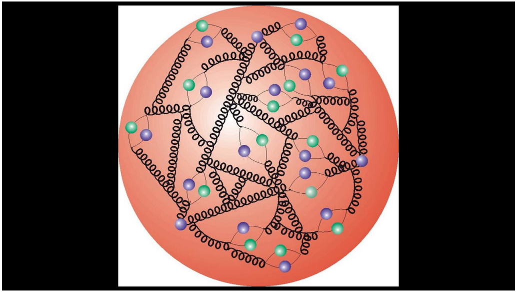
But once we start studying potentials with more complicated symmetries, e.g. $\mathbb{Z}_2 \times \mathbb{Z}_2$, then anomalies are perfectly allowed, e.g.

$$H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$$

So a non-degenerate symmetric ground state is probably the exception rather than the rule.

- This argument that we used, i.e. that the anomaly cannot disappear as long as the corresponding symmetries are not broken, is an example of the general philosophy of 't Hooft.
- When 't Hooft anomalies are present, the ground state is necessarily nontrivial since it needs to "match" the anomalies and lead to a non-gauge invariant partition function under gauge transformations of the background fields.
- It appears that such anomalies are much more common than traditionally thought. Such anomalies exist in all dimensions and they do not require fermions in the Lagrangian.

This example that we discussed turns out to be very educational because almost all the aspects that we discussed carry over to Yang-Mills Theory!



First let us review the classical Yang-Mills theory with $SU(N)$ gauge group. The dynamical variable, a, is a space-time vector which is also traceless Hermitian $N \times N$ matrix and we define

$$F = da + i[a, a] \ .$$

We write the classical action/energy functional

$$S = \int d^4x Tr(F \wedge \star F) = \int d^4x |F|^2 \ .$$

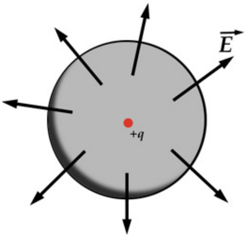
The Euler Lagrangian equations, known as the Yang-Mills equations, are very nonlinear

$$d_a F = d_a \star F = 0 \ ,$$

where $\star$ is the Hodge dual.

There is of course a huge amount of work on these classical PDEs.

This theory is a nonlinear version of Maxwell's theory. The crucial physical difference is that the fields E, B in Maxwell's theory do not carry electric charge. This is why it is linear. Here E and B carry charge. This leads to an important difference. In Maxwell's theory we have a Gauss Law



we can extract the total charge by measuring

$$\int_2 E$$

But in Yang-Mills theory we cannot quite do that because the field itself is charged so it is hard to isolate the probe charge. So imagine that we put an external probe particle in the representation R of $SU(N)$. The fields E, B are in the representation $Adj(SU(N))$ and hence they may confused us and we can measure any charge of the form

$$R \otimes Adj \otimes Adj \cdots \ .$$

So we cannot measure R but we can measure it N-ality. We thus have a Z_N "center" symmetry. In Maxwell's theory it is $SO(2)$ because we can measure any $U(1)$ charge.

For example, in $SU(2)$ Yang-Mills, we can measure whether R is even- or odd- dimensional representation but not more than that.

The quantum theory is, however, non-unique! The physical reason is as before – in the various possible histories, there is a hidden internal circle and the histories can be divided according to how many times this circle has been wound around.

$$\frac{1}{2\pi}\int dx\ \dot{q} \longleftrightarrow \frac{1}{8\pi^2}\int d^4x\ Tr(F\wedge F)$$

Therefore in the quantum theory we have as before a θ parameter which we can add to the action

$$\frac{i\theta}{8\pi^2}\int d^4x Tr(F \wedge F)$$

and it is manifest that $\theta \simeq \theta + 2\pi$.

The two choices $\theta = 0, \pi$ are special! They preserve time-reversal symmetry. Or equivalently, they preserve CP symmetry. This is again slightly nontrivial at $\theta = \pi$ since we need to use the fact that $\theta = \pi$ and $\theta = -\pi$ describe the exact same theory. But there is, as before, a nontrivial similarity transformation involved at $\theta = \pi$.

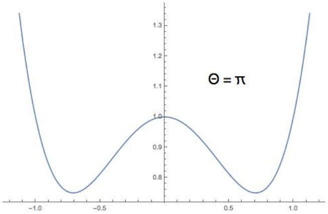
Now we need to discuss the analog of the $SO(2)$ symmetry, $q \to q + \alpha$. This is a nontrivial part of the story. This role is played by the center symmetry. In Maxwell's theory it is $U(1)$ and in $SU(N)$ Yang-Mills theory it is $\mathbb{Z}_N$.

The symmetry group at $\theta = 0, \pi$ is therefore

$$\textit{time} - \textit{reversal} \times \mathbb{Z}_N$$

At $\theta = 0$ there is no anomaly and for $\theta = \pi$ there is an anomaly.

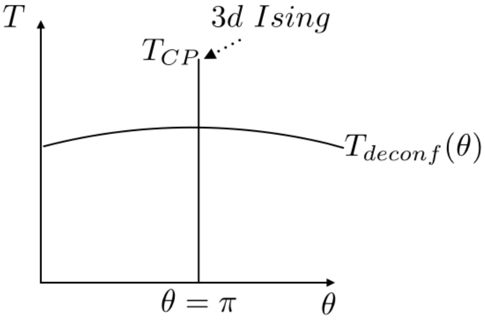
The model at $\theta = 0$ is gapped with a trivial confining vacuum, as observed by lattice simulations. The model at $\theta = \pi$ however cannot have a trivial ground state. One way to saturate the anomaly is to **break time reversal spontaneously** and have two degenerate confining ground states.



As we crank up the temperature we expect time reversal would be eventually restored. Also, we expect the gluons would be liberated. The anomaly can be saturated only if the latter happens before the former.

Therefore, at sufficiently large N (probably $N \geq 3$) we expect the following phase diagram for Yang-Mills theory

$$T_{CP} \geq T_{deconf}(\theta = \pi)$$



Let us now briefly discuss the question of whether this mechanism operates in nature. First of all, in QCD there are fundamental quarks and the $\mathbb{Z}_N$ is thus not present. But there is a symmetry that rotates between the quarks which in some sense replaces it. If one of the quark masses is negative, e.g. if $m_u < 0$ then one may have spontaneously broken time reversal invariance very similar to what we described. I believe we do not actually know whether this happens.

There could be an additional sector to the Standard Model where the above mechanism occurs. Perhaps this could trigger the violation of CP that we see in the real world.

We saw that using anomalies and topology one can establish a non-perturbative result about Yang-Mills theory. But there are many other examples where we can make striking predictions using new discrete 't Hooft anomalies, for example,

- QCD (the center symmetry is not necessary!)
- Néel-VBS transition and its $N_f > 2$ generalizations.
- Chern-Simons theories.
- Abelian Higgs model in two dimensions (of which the Haldane model is a special case).

In some cases we can also make strong claims about the phase diagram at finite temperature, as we did in Yang-Mills theory.

Summary

- Topology offers a window into non-perturbative physics, often leading to concrete, verifiable predictions.
- Anomalies can exist in bosonic systems, even in 2+1 dimensions (and also 0+1 and 4+1).
- The mathematical underpinnings of these developments lie in understanding discrete characteristic classes of principle bundles (and also higher gauge theory).
- There are applications for domain walls, thermal physics (with and without chemical potentials), the nature of various transitions (1st/2nd order), and testing dualities.