The Universe in a Matrix: Large N gauge theories, matrix models and non-commutative space-time

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1

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A fascinating field theory I

SU(N) Yang-Mills theory in d dimension (d=4)

Euclidean Action:

$$S = \frac{1}{2g^2} \int dx \, \mathrm{Tr}(F_{\mu\nu}(x) F_{\mu\nu}(x))$$

where $F_{\mu\nu}(x) = i[D_\mu, D_\mu]$ with $D_\mu = \mathbf{I}\partial_\mu - iA_\mu(x)$ the covariant derivative. $(A_\mu(x) \in su(N))$

Invariant under gauge transformations.

It simply defined but non-trivial QFT that

- Sits at the core of particle interactions in Nature
- Has no free parameters
- Expected to be well-defined by itself(UV complete)
- Exhibiting many non-trivial features

A fascinating field theory II

- Asymptotic freedom $\lim_{\mu \longrightarrow \infty} g_R(\mu) \longrightarrow 0$
- Dimensional transmutation (generates a mass scale quantum mechanically) Λ_{QCD}
- Non-trivial spectrum (glueballs) and a mass gap M and M_i
- Confinement $E(r) = \sigma r \quad (\sigma$ string tension)
- Finite temperature Phase transition T_c
- Topological charge and susceptibility χ
- **With quarks added**: Chiral symmetry breaking, meson spectrum, U(1) problem

How to compute all these quantities?

Path-integral formulation:

$$Z = \prod_{\mu,x} \int \mathcal{D}A_\mu(x)\ e^{-S}$$

Not a well-defined Mathematical object.

Perturbation Theory: One can set up a calculational procedure by expanding in powers of g^2.

Observables involving short-distances can be computed.

Most of the mass scales presented previously are zero to all orders in PT.

Clay Institute Millenium Problem

The lattice formulation

Space-Time $\Rightarrow$ hypercubic lattice $\mathcal{L}$

Dynamical variables: SU(N) matrices $U_\mu(n) \equiv U(l)$ (l=link)

Given a loop $\mathcal{C}$:

$$U(\mathcal{C}) = T \prod_{l \in \mathcal{C}} U(l)$$

The partition function:

$$Z_L = \prod_l \left(\int dU(l) \right) \; e^{-S_L}$$

where the simplest action S_L (Wilson action) is

$$S_L = -\frac{1}{g_L^2} \sum_{P \in \mathrm{plaquettes}} \mathrm{Tr}(U(P))$$

Main observables $W(\mathcal{C}) = \frac{1}{N} \langle \mathrm{Tr}\, U(\mathcal{C}) \rangle$

$1/N$ expansion

- An unexpected small parameter found by 't Hooft: $1/N$
- One must scale the coupling keeping $\lambda = g^2 N$ constant.
- In Perturbation Theory the $1/N^2$ expansion corresponds to an expansion in the genus of the surface in which the diagrams can be drawn.
- The leading term is the **large N theory**. Only planar diagrams survive.

The large N theory is a simpler theory sharing most of the non-trivial properties of finite N. Sits at the crux of the connection between string theory and gauge theories

The large N limit of QCD

Lowest term in the expansion

Properties	
Asymptotic freedom	
Dimensional transmutation	
Confinement	
Chiral Symmetry breaking	
Chiral P.T.	
Topological charge	
U(1) problem	
Glueball spectrum (Mass gap)	
Meson spectrum	
AdS/CFT correspondance?	
Difficult To solve	

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Properties	**Simplification**
Asymptotic freedom $\surd$	Only planar diagrams
Dimensional transmutation $\surd$	No scale theory
Confinement $\surd$	Factorization
Chiral Symmetry breaking $\surd$	No dynamical quarks (Quenched)
Chiral P.T. $\surd$	No chiral logs
Topological charge $\surd$	No instantons
U(1) problem $\surd$	$m_{\eta'} \longrightarrow 0$
Glueball spectrum (Mass gap)$\surd$	Stable-No mixing
Meson spectrum $\surd$	Stable + No mixing
AdS/CFT correspondance? $\surd$	Free strings/Classical gravity
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Difficult To solve $\surd$	**OOPS!**

The twisted Eguchi-Kawai model TEK

$$Z = \prod_\mu \left(\int dV_\mu \right) e^{-S_{\rm TEK}}$$

with

$$S_{\rm TEK} = -\frac{N}{\lambda_L} \sum_{\mu,\nu} z_{\mu\nu} {\rm Tr}(V_\mu V_\nu V_\mu^\dagger V_\nu^\dagger)$$

and $z_{\mu\nu} = z^*_{\nu\mu} = e^{2\pi i n_{\mu\nu}/N}$.

- This matrix model follows by reducing to 1 point the lattice gauge theory on a torus with twisted boundary conditions. Volume Independence
- The choice of the antisymmetric twist tensor $n_{\mu\nu} \in (Z/NZ)^d$ is crutial (see later).

CLAIM:

$$\lim_{N \longrightarrow \infty} \prod_P z(P) \langle {\rm Tr}(V(\mathcal{C})) \rangle \Longrightarrow W(\mathcal{C})$$

Explanation of the equivalence (OLD RESULTS)

- First proof of equivalence (Eguchi and Kawai 1982): equality of the Schwinger-Dyson equations satisfied by loops. (Valid for all $n_{\mu\nu}$).
  Assumes invariance under **center symmetry**: $V_\mu \longrightarrow z_\mu V_\mu$ with $z_\mu \in Z_N$.
  Is this symmetry broken spontaneously?
- **Weak coupling analysis** (small λ_L): Symmetry is broken unless the twist tensor is irreducible.
  In 4D one can choose the symmetric twist $N = \hat{L}^2$ and $|n_{\mu\nu}| = k\hat{L}$ with $\gcd(k, \hat{L}) = 1$. The minimum action configuration ($V_\mu = \Gamma_\mu$), where
  $$\Gamma_\mu \Gamma_\nu = e^{2\pi i n_{\mu\nu}/N} \Gamma_\nu \Gamma_\mu$$
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- **What is its physical interpretation?**
  Space-time degrees of freedom are embeded in the group, but **HOW?**

Perturbation theory for the TEK model

Beyond lowest order: $V_\mu = e^{-ig_L A_\mu} \Gamma_\mu$
Quadratic piece of the action is:

$$\mathrm{Tr}(\delta_\mu A_\nu - A_\nu - \delta_\nu A_\mu + A_\mu)^2$$

where

$$\delta_\mu \Phi \equiv \Gamma_\mu \Phi \Gamma_\mu^\dagger$$

Crucial ingredient: A nice basis of the Lie algebra

$$\lambda^a \longrightarrow \lambda(\vec{p}) \quad / \qquad \delta_\mu \lambda(\vec{p}) = e^{ip_\mu} \lambda(\vec{p})$$

with $\vec{p} = 2\pi \vec{n}/\hat{L}$ **Colour Momenta**

Propagator is the same as in an $\hat{L}^4$ lattice

Finite N corrections in propagators look like finite volume corrections.

Perturbation theory for the TEK model

Feynman rules for Vertices:

$$f_{abc} \longrightarrow f(\vec{p},\vec{q},\vec{l}) \propto \delta(\vec{p}+\vec{q}+\vec{l}) \exp\{i\hat{L}\bar{k}\tilde{\epsilon}_{\mu\nu}\, p_\mu q_\nu/(4\pi)\} - (\vec{p} \leftrightarrow \vec{q})$$

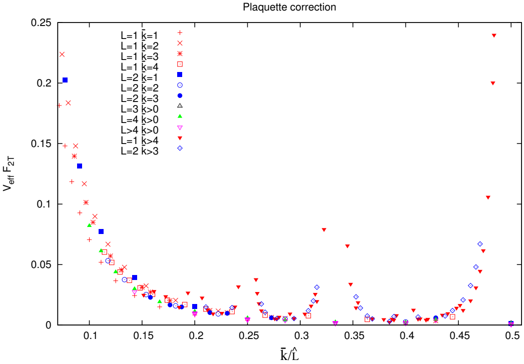
with $\bar{k}k = 1 \bmod N$ and $\tilde{\epsilon}\epsilon = \mathbf{I}$.

- Colour Momentum conservation at the vertices
- Overall phase absent for planar diagrams.
- Non-planar diagrams killed by rapidly oscillating phases as $\hat{L} = \sqrt{N} \longrightarrow \infty$

Recent calculation (*Garcia Perez, GA, Okawa 2017*) of Wilson loops in L^4 lattice to order λ^2 shows the rate of vanishing.

Non Planar contribution to order λ^2

The non-planar contribution $\delta \hat{W}_{NP}$ goes to zero as $1/(L\hat{L})^4$ with a coefficient depending on $\bar{k}/\hat{L}$:



Notice that the choice of $\bar{k}$ and k affects the corrections.

Non-Commutative Field Theories I

- A. Connes introduced the notion of Non-Commutative space. This is induced by generalizing the *commutative* algebra of functions on space.
- This later developped into gauge theory defined in these spaces (*Connes Rieffel 1987*)
- Appeared as a special limit of strings (*Seiberg-Witten 1999*)
- Action and Feynman rules coincide with those that we had obtained by taking the continuum version of the TEK model (*GA, Korthals-Altes 1983*):
  $\delta(q+k+l)exp\{-i\theta_{\mu\nu}q^\mu k^\nu/2\}$ at the vertices.
- New phenomena appeared in the computation of loop integrals (UV-IR mixing *Minwalla, Van Raamsdonk, Seiberg* 2000)

Non-Commutative Field Theories II

- On the non-commutative torus for rational values of the *dimensionless* non-commutative parameter $\bar{\theta}_{\mu\nu} = \bar{n}_{\mu\nu}/N$, the system is equivalent to U(N) gauge theory with TBC (Morita duality).
- It was proposed to use the TEK model as a lattice regularization of non-commutative Yang-Mills (*Ambjorn et al 2000*).
- The issue of continuity in $\bar{\theta}$ was put forward (*Barbon, Alvarez-Gaume 2001*). Is it possible to define the theory at irrational $\bar{\theta}$ as a limit of rationals?
- The finite torus size l_μ eliminates the infrared singularity, but still the self-energy becomes negative and could give rise to singularities at finite values of the coupling: **Tachyonic instabilities** (*Hakayama, Guralnik et al*).

Is the matrix model equivalence valid?

For the equivalence to survive the continuum limit it should hold in the scaling region $Ma_L(\lambda_L) \ll 1$ but for large effective sizes $Ma_L(\lambda_L)\sqrt{N} \gg 1$

Cracks in the wall

- ♠ The potential problems associated with *Tachyonic Instabilities*
- ♠ Signs of center symmetry breaking observed in numerical studies of TEK at larger N. *Ishikawa-Okawa(2003), Teper-Vairinhos(2007)*
- ♣ Condensation observed in numerical results using TEK as a non-perturbative definition of NC field theory. *(Bietenholz et al 2006)*

All problems avoided if $k/\sqrt{N}$ and $\bar{k}/\sqrt{N}$ kept bigger than a certain value in the large N limit (*GA-Okawa 2010*).

Testing the equivalence

- No symmetry breaking observed up to $N = 1369 = 37^2$
- Direct test of equivalence on the lattice(*AGA, Okawa 2014*):
  a) Measuring Wilson loops in a big lattice ($L^4 = 16^4$) with periodic boundary conditions and various N ($N = 8 - 16$) and extrapolating the results to infinite N (2nd degree polynomial in $1/N^2$).
  b) Measure the loops on the matrix model.
  For the plaquette:

$\lambda_L = 1/0.36$ extrapolated		# dofs=1.7 10^7
TEK $N = 289$		# dofs= 0.8 10^5

The same happens for other loops and at other couplings.

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  We also measured the string tension in both theories:
  N=3,5,6,8 32^4 PBC lattice **0.515(3)**
  L=1 N=841 TBC **0.513(6)**

Yang-Mills in $T_2 \times \mathbb{R}$

- Hamiltonian picture: $l \times l$ spatial torus with twist $n_{\mu\nu} = k\epsilon_{\mu\nu}$.
- Center symmetry is now Z_N^2. States are labelled by IRREP: $\vec{e} \in (\mathbb{Z}/N\mathbb{Z})^2$. **electric flux**
- Dependence of lowest Energy states on the parameters:

$$E(\vec{e}, N, k, l, \lambda) = \lambda \mathcal{E}(\vec{e}, N, k, x)$$

with $x = Nl\lambda/(4\pi)$.

What do we know and expect?

For small sizes ($l\lambda \ll 1$) Perturbation Theory is a good approximation.
The leading order is a free gluon gas $\mathcal{E} \sim |\vec{n}|/2x$ where $\vec{n} = N||k\vec{e}_T/N||$

Yang-Mills in $T_2 \times \mathbb{R}$

At the next order in λ the main contribution is the self-energy term:

$$\mathcal{E}^2(\vec{e}, N, k, x) = \frac{|\vec{n}|^2}{4x^2} - \frac{G(\vec{e}, N)}{x} \sim \frac{|\vec{n}|^2}{4x^2} - \frac{1}{16\pi^2 x |\vec{e}/N|^2} + R$$

G is positive and predicts a *tachyonic instablity* at $x = x_c$

Can one avoid the singularity by tuning k? $|n| \nearrow \Rightarrow x_c \nearrow$

At large volumes $\lambda l \gg 1$ one expects the **CONFINEMENT** behaviour

$$\mathcal{E} \to \frac{\sigma(\vec{e}, N, k, x)}{\lambda} l = 4\pi \frac{\sigma}{\lambda^2} x \chi(\vec{e}/N)$$

where $\chi(x) \sim x$ determines the k-string spectrum. $\frac{\sigma}{\lambda^2} = \frac{\tau}{8\pi}$.
The correction to this term (Luscher term) is an x independent constant added to $\mathcal{E}^2$ (exact in Nambu-Goto)

Yang-Mills in $T_2 \times \mathbb{R}$

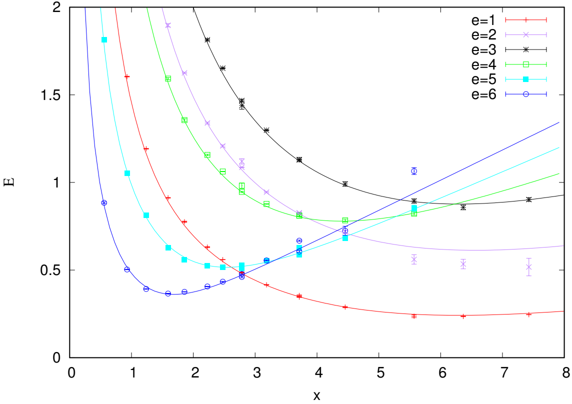
What happens at intermediate values of λl? Are there tachyonic instabilities?

Analyze the problem by a lattice simulations exploring all dependencies: $L = 1, \ldots 28$, $N = 5, 7, 11, 13, 17, 34, 89$ and many k values and many values of λ_L ($x \in [0.2, 7]$)

RESULTS *Garcia-Perez,GA,Koren,Okawa 2013,2018*

- Continuity in $\bar{\theta} = \bar{k}/N$ for n fixed.
- Behaviour at intermediate values well described by analytic function obtained by adding the perturbative and confining terms $F(nx, Z(n, k, N))$, where $Z = n||n\bar{k}/N||$.
- In the $e = 0$ sector (glueball) for $x > 3$ Torelon-torelon states (states of opposite electric fluxes), coexist with a new state having constant energy $M/\lambda \sim 0.85$ and coupling mostly to Wilson loops.

Size dependence of energies $N = 17$ $k = 3$



Consequences

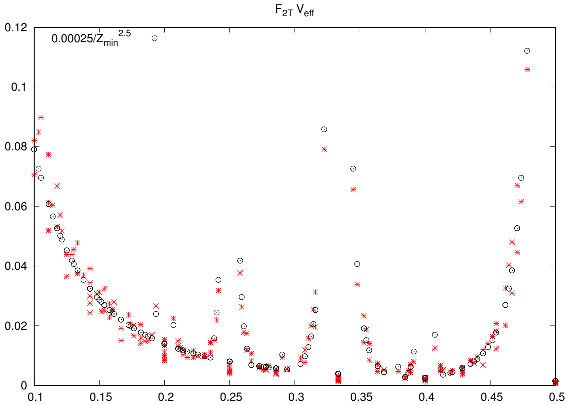
Having an analytic expression for the energies allows us to explore its behaviour at large N. The minimum energy is a function of Z: $\mathcal{E}_{\rm min}(n) = \phi(Z) \sim A(Z - 0.1)$. Thus the condition not to have tachyonic instabilities is

$$\min_n \mathcal{E}_{\rm min}(n) > 0 \quad \Rightarrow \quad Z_{\rm min} \equiv \min_n n||n\bar{k}/N|| > 0.1$$

This leads to the following conclusions *Chamizo GA 2017*

- Can one choose a k for any N without tachyonic instabilities? $\Leftrightarrow$ Zaremba conjecture (1974).
- For almost any N $Z_{\rm min} > 1/7$ *Huang 2015* .
- The best sequence that maximizes the minimal energy is given by the Fibonacci numbers: $N = F_p$ and $k = \bar{k} = F_{p-2}$
- The set of irrationals θ such that $\bar{k}_p/N_p \longrightarrow \theta$ and has no T.I. is a set of Haussdorff dimension $d_H \sim 0.7 < 1$ (N.C.)
- This seems to extend to defining a subleading continuous non-planar correction to Wilson loops in 4 dimensions.

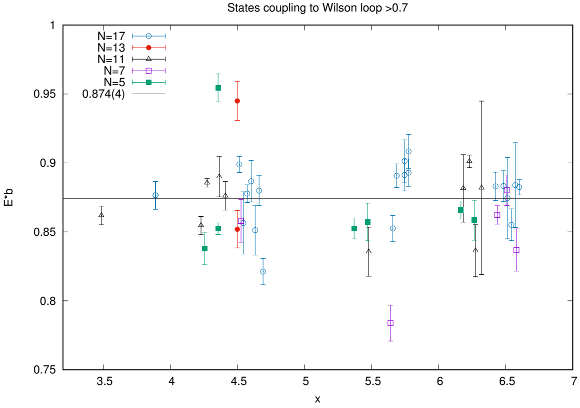
Non-planar correction to plaquette



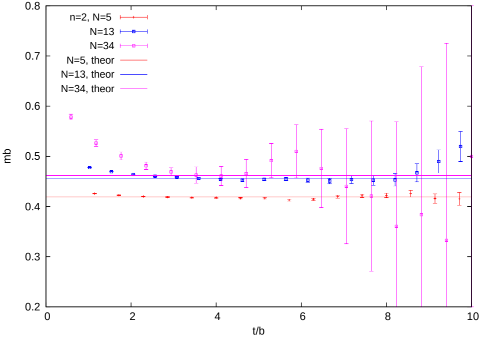
Conclusions

- TEK model provides a way to study Yang-Mills at large N which is at least competitive with extrapolations. List of possible observables: Meson spectrum, Finite T_c, condensate, glueball spectrum(?)
- The approach to infinite N is connected to theories in Non-commutative space. The choice of flux k is non-trivial. $Z_{\rm min}(\bar{k}, N)$ plays a crucial role.
- The twisted reduction mechanism allows Matrix models for many other interesting theories at large N: Adjoint-QCD with quarks in the adjoint, QCD in the Veneziano limit, Principal chiral models. No competitive extrapolation for theories with dynamical fermions.
- Many things are yet to be clarified at the theoretical level. Supersymmetric extensions remain a challenge.

Glueball mass



Comparisons Fibonacci



N-dependence of plaquette e.v.

We studied the plaquette ($R = T = 1$) at $b = 0.36$ at various L and N. This is a very precise quantity (errors 10^{-5}).

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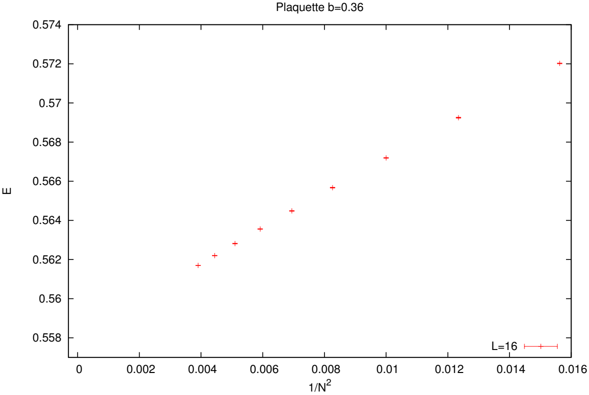

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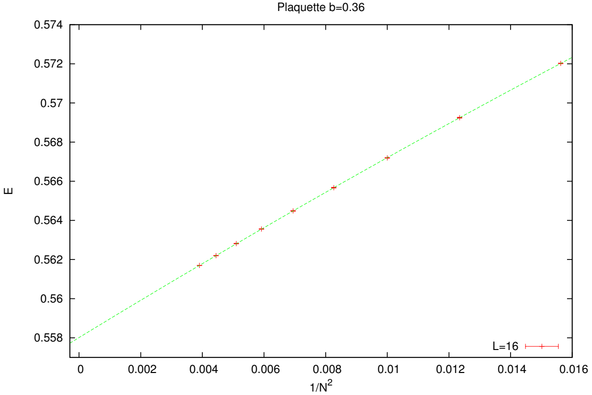

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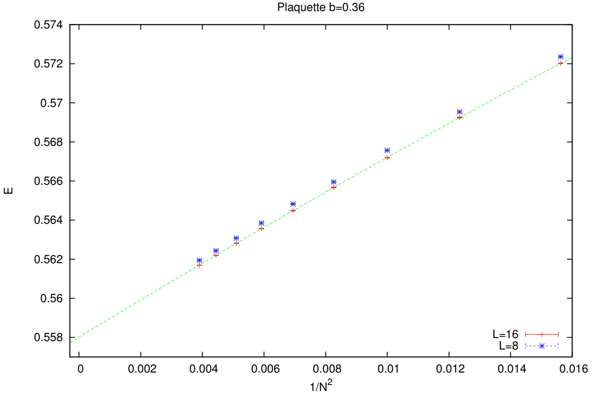

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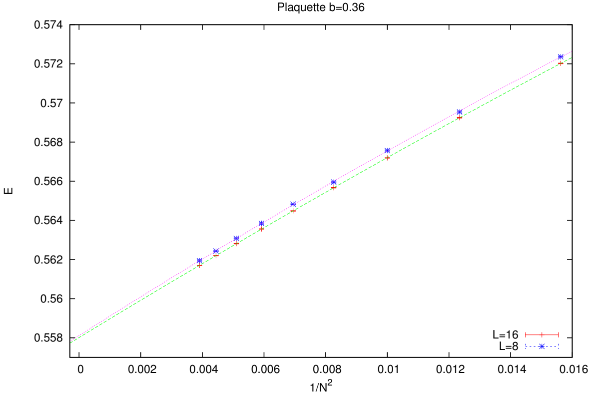

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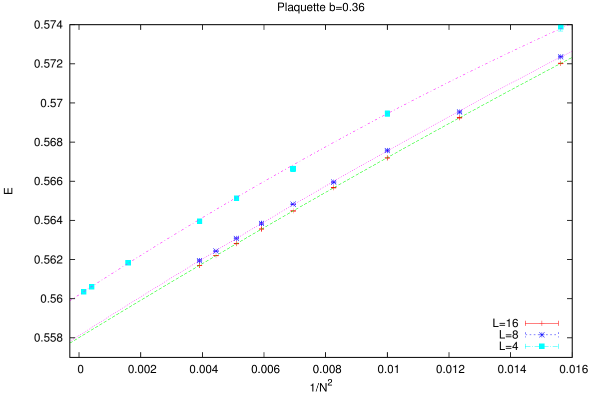

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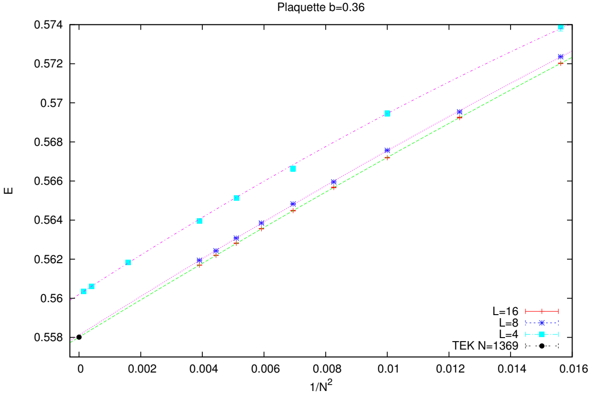
And twisted boundary conditions $k \neq 0$ (TEK $L = 1$)



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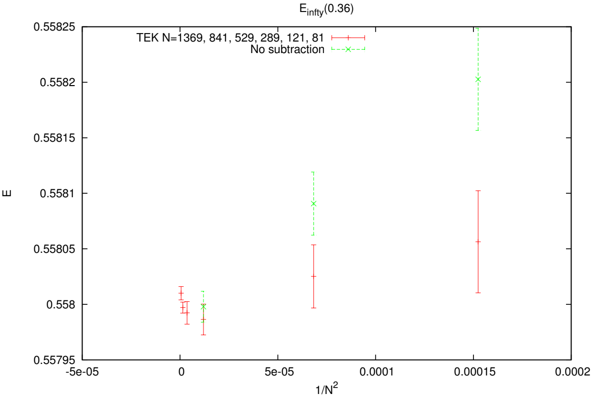

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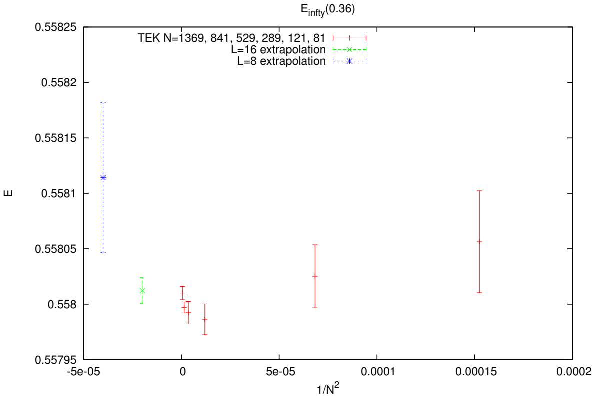
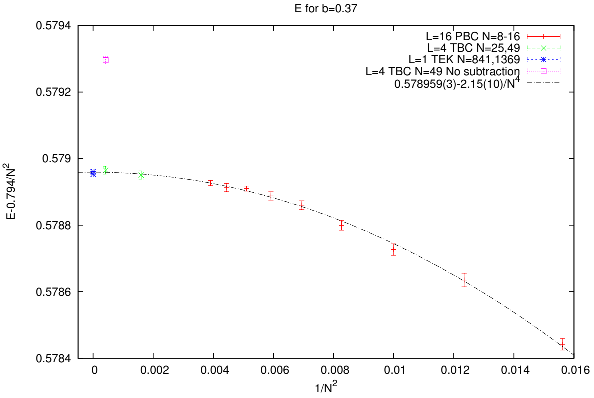


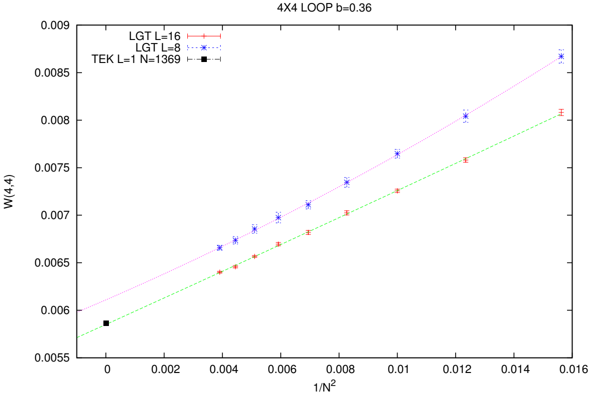
Other values of b and R

The result extends to other values of b:
Example $b = 0.37$: The $1/N^2$ is approximately universal

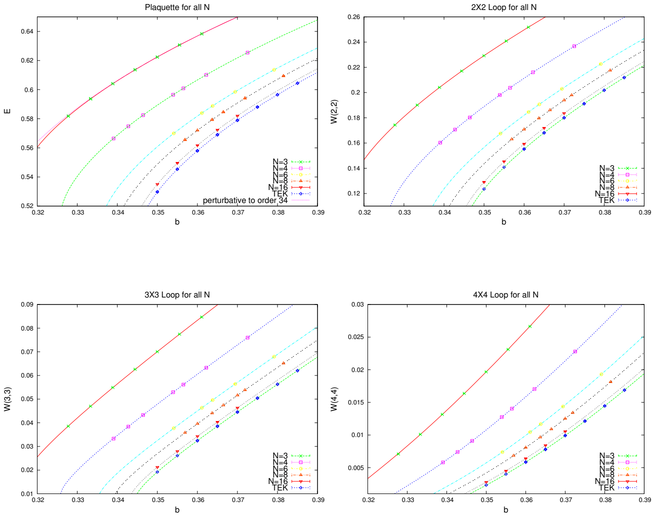


Other values of b and R

The same is true about other Wilson loops $R = 2, 3, 4$



A final view


Plaquette for all N
E
b
N=3
N=4
N=6
N=8
N=16
TEK
perturbative to order 34
2X2 Loop for all N
W(2,2)
b
N=3
N=4
N=6
N=8
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TEK
3X3 Loop for all N
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N=3
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TEK
4X4 Loop for all N
W(4,4)
b
N=3
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