Primordial Non-Gaussianity in Heavy-Ion Collisions

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Based on

arXiv:1904.10350 = PRC **100** (2019) 014909
RSB, G. Giacalone and J.-Y.Ollitrault

and

arXiv:1811.00837 = PRC **99** (2019) 014907
RSB, G. Giacalone and J.-Y.Ollitrault

Introduction

- Collective Flow: one of the most important diagnostic tools to probe
  - the initial state of the system
  - properties of the QGP
- Initial years: mainly directed (v_1) & elliptic (v_2) flows
- Recent years: $v_n\,(n = 1\text{-}8)$, $v_n\{m\}\,(m = 2, 4, 6, 8)$, symm. cumulants, event-plane correlators, nonlin. response coeffs, $P(v_n)$, non-Gaussianities of flow fluct
- Data on cumulants $\Rightarrow$ non-Gaussian statistics of the energy-density field created right after the collision.
- Cosmology: primordial non-Gaussianity $\approx$ zero.

Moments of a Probability Density Function (pdf)

n-th moment of a (real, continuous) function $f(x)$, about a constant a:

$$\mu_n(a) \equiv \int_{-\infty}^{\infty} (x-a)^n f(x) dx.$$

Let $f(x)$ be the pdf, normalized to unity. Usually, MOMENT (μ'_n): $a = 0$, CENTRAL MOMENT (μ_n): $a = \mu \equiv \langle x \rangle$. Note $\mu'_0 = 1 = \mu_0$, $\mu'_n = \langle x^n \rangle$.

$$\begin{aligned} \mu_1 &= 0, \\ \mu_2 &= \left\langle x^2 \right\rangle - \mu^2 \equiv \text{VARIANCE } (\sigma^2) \equiv \text{ (std. dev. } \sigma)^2. \end{aligned}$$

NORMALIZED CENTRAL MOMENTS (scale inv. or dim. less) μ_n/σ^n

$$\begin{aligned} \mu_1/\sigma &= 0, \\ \mu_2/\sigma^2 &= 1, \\ \mu_3/\sigma^3 &\equiv \text{SKEWNESS } (\gamma), \\ \mu_4/\sigma^4 &\equiv \text{KURTOSIS } (\kappa). \end{aligned}$$

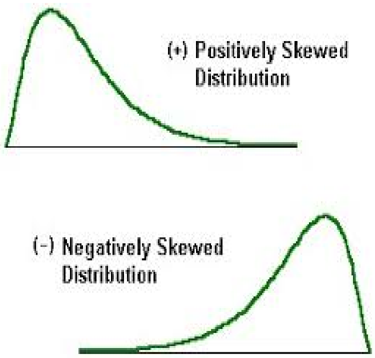
- Variance: a measure of the spread of the random numbers about their mean value.

- Skewness: a measure of the lopsidedness or asymmetry of the distribution about its mean. If the left (right) tail is drawn out, or in other words, is longer than the right (left) tail, the distribution is said to be left(right)-skewed and has a negative (positive) skewness.

- Kurtosis: a measure of the outliers or the heaviness of the tails of the distribution as compared to the normal distribution with the same variance. A nonnegative number. Excess kurtosis $\equiv (\kappa - 3)$ may be $\pm$ve.

Measures of Non-Gaussianity of Fluctuations

Skewness



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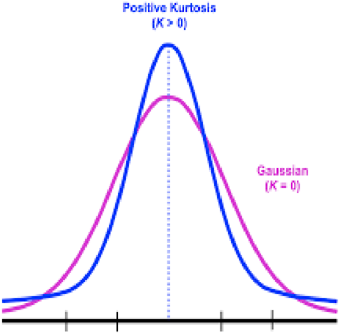
Excess Kurtosis



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Correlation Functions ρ and Cumulants C

$$
\begin{aligned}
\rho(1) &= C(1), \\
\rho(1,2) &= \rho(1)\rho(2) + C(1,2), \\
\rho(1,2,3) &= \rho(1)\rho(2)\rho(3) + \rho(1)C(2,3) + \rho(2)C(3,1) \\
&+ \rho(3)C(1,2) + C(1,2,3), \\
&\equiv \rho(1)\rho(2)\rho(3) + \sum_{(3)} \rho(1)C(2,3) + C(1,2,3), \\
C(1) &= \rho(1), \\
C(1,2) &= \rho(1,2) - \rho(1)\rho(2), \\
C(1,2,3) &= \rho(1,2,3) - \sum_{(3)} \rho(1)\rho(2,3) + 2\rho(1)\rho(2)\rho(3).
\end{aligned}
$$

C: True or genuine correl. Unlike ρ, C vanishes if any one or more particles is statistically indep of the others. The n-particle cumulant measures the statistical dependence of the entire n-particle set. C are also called Connected Correlation Functions.

It is a model in which particles are emitted randomly and independently according to some underlying probability distribution in each event. The azimuthal probability distribution that fluctuates event to event is:

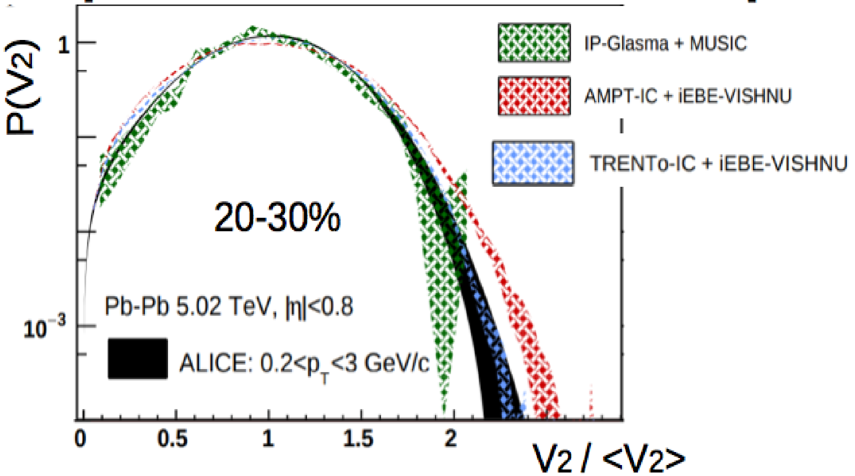
$$P(\phi) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} V_n(p_T, \eta) e^{in\phi} = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} v_n e^{in(\phi - \Psi_n)}$$

V_n: (complex) flow in the azimuthal or transverse plane, v_n: magnitude, Ψ_n: event-plane angle. Equivalently,

$$\frac{dN}{d\phi} = \frac{N}{2\pi} \left(1 + \sum_{n=1}^{\infty} 2 v_n \cos n(\phi - \Psi_n) \right)$$

Probability Distribution Function of Elliptic Flow

[ALICE Collaboration, arXiv 1804:02944]



ALICE data on $P(v_2) \cdot \langle v_2 \rangle$ compared with various models.

Flow Determination using Multiparticle Correlations

Two-, four-. six-, eight-particle azimuthal correlations:

$$\langle\langle 2\rangle\rangle = \langle\langle e^{in(\phi_1-\phi_2)}\rangle\rangle,$$
$$\langle\langle 4\rangle\rangle = \langle\langle e^{in(\phi_1+\phi_2-\phi_3-\phi_4)}\rangle\rangle,$$
$$\langle\langle 6\rangle\rangle = \langle\langle e^{in(\phi_1+\phi_2+\phi_3-\phi_4-\phi_5-\phi_6)}\rangle\rangle,$$
$$\langle\langle 8\rangle\rangle = \langle\langle e^{in(\phi_1+\phi_2+\phi_3+\phi_4-\phi_5-\phi_6-\phi_7-\phi_8)}\rangle\rangle.$$

$\langle\langle\cdots\rangle\rangle$: averaging over all multiplets in a single collision event and then over all events in a given centrality class.

Note that all these "observables" are invariant under rotation in the azimuthal plane, as they should be.

Flow using Multiparticle Correlations (contd.)

Multiparticle cumulants: ... [Borghini et al. nucl-th/0105040]

$$c_n\{2\} = \langle\langle 2\rangle\rangle,$$
$$c_n\{4\} = \langle\langle 4\rangle\rangle - 2\langle\langle 2\rangle\rangle^2,$$
$$c_n\{6\} = \langle\langle 6\rangle\rangle - 9\langle\langle 2\rangle\rangle\langle\langle 4\rangle\rangle + 12\langle\langle 2\rangle\rangle^3,$$
$$c_n\{8\} = \langle\langle 8\rangle\rangle - 16\langle\langle 2\rangle\rangle\langle\langle 6\rangle\rangle - 18\langle\langle 4\rangle\rangle^2 + 144\langle\langle 2\rangle\rangle^2\langle\langle 4\rangle\rangle - 144\langle\langle 2\rangle\rangle^4.$$

In the absence of nonflow correlations:

$$c_n\{2\} = \langle v_n^2\rangle,$$
$$c_n\{4\} = \langle v_n^4\rangle - 2\langle v_n^2\rangle^2,$$
$$c_n\{6\} = \langle v_n^6\rangle - 9\langle v_n^2\rangle\langle v_n^4\rangle + 12\langle v_n^2\rangle^3,$$
$$c_n\{8\} = \langle v_n^8\rangle - 16\langle v_n^2\rangle\langle v_n^6\rangle - 18\langle v_n^4\rangle^2 + 144\langle v_n^2\rangle^2\langle v_n^4\rangle - 144\langle v_n^2\rangle^4.$$

Flow using Multiparticle Correlations (contd.)

Finally, flow coefficients in terms of cumulants:

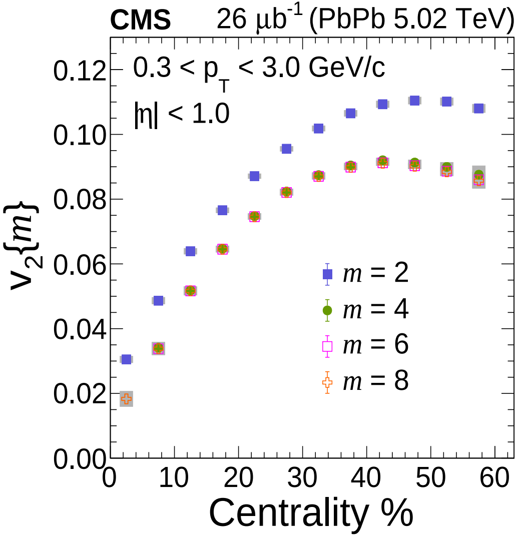
$$v_n\{2\} = \sqrt{c_n\{2\}},$$

$$v_n\{4\} = \sqrt[4]{-c_n\{4\}},$$

$$v_n\{6\} = \sqrt[6]{\frac{1}{4}c_n\{6\}},$$

$$v_n\{8\} = \sqrt[8]{-\frac{1}{33}c_n\{8\}}.$$

Explanation: If the magnitude of the flow vector does not fluctuate event-to-event, then $c_n\{2\} = v_n^2 > 0$, $c_n\{4\} = -v_n^4 < 0$, $c_n\{6\} = 4v_n^6 > 0$, $c_n\{8\} = -33v_n^8 < 0$. Each of the above equations would then give v_n as expected.



CMS 1711.05594

$v_2\{2\} > v_2\{4\} \approx v_2\{6\} \approx v_2\{8\}$

Consistent with the Gaussian ansatz by

Voloshin et al. 0708.0800

$P(v_x, v_y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(v_x-\bar{v})^2+v_y^2}{2\sigma^2}\right)$ implies

$v_2\{2\} = \sqrt{\bar{v}^2 + 2\sigma^2}$ and $v_2\{4\} = v_2\{6\} = v_2\{8\} = \bar{v}$. BUT, ⋯

Evidence of Non-Gaussianity in Elliptic Flow Fluctuations

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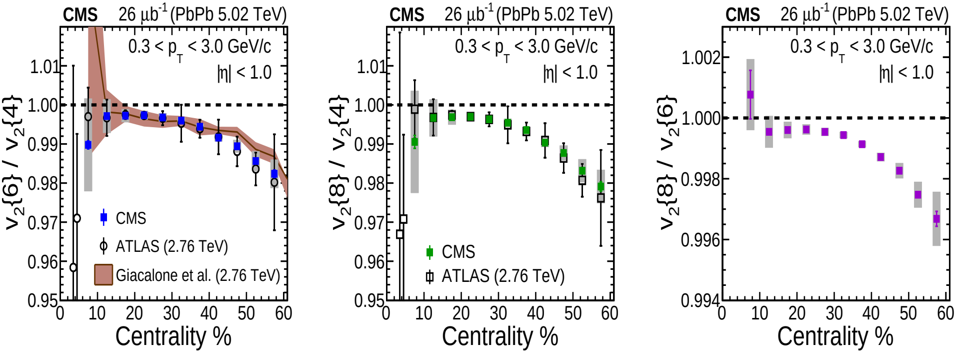
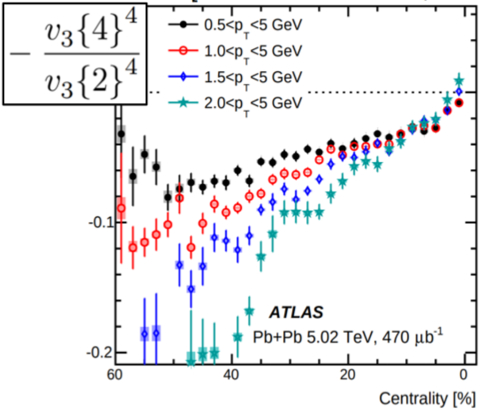


Evidence of Non-Gaussianity in Triangular Flow Fluct.

Gaussian ansatz $P(v_x, v_y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{v_x^2+v_y^2}{2\sigma^2}\right)$ implies $v_3\{2\} = \sqrt{2}\sigma$ and $v_3\{4\} = v_3\{6\} = v_3\{8\} = 0$. BUT, $\cdots$

[ATLAS Collaboration, 1904.04808]



Recall

$$-\frac{v_3\{4\}^4}{v_3\{2\}^4} = \frac{\langle v_3^4 \rangle}{\langle v_3^2 \rangle^2} - 2$$

Measure of (excess) kurtosis of triangular flow fluctuations

PRESENT WORK

Perturbative Expansion of Initial Anisotropy ε_n

$\mathbf{s} = (x, y)$, $\mathbf{s}_0 =$ centre of the distribution of density $\rho(s)$

$$\varepsilon_n = \frac{\int_{\mathbf{s}} (\mathbf{s} - \mathbf{s}_0)^n \rho(\mathbf{s})}{\int_{\mathbf{s}} |\mathbf{s} - \mathbf{s}_0|^n \rho(\mathbf{s})} \text{ where } \mathbf{s}_0 \equiv \frac{\int_{\mathbf{s}} \mathbf{s}\rho(\mathbf{s})}{\int_{\mathbf{s}} \rho(\mathbf{s})} = \frac{\int_{\mathbf{s}} \mathbf{s}\delta\rho(\mathbf{s})}{\int_{\mathbf{s}} \rho(\mathbf{s})}$$

Expansion of ε_n in powers of density fluctuations $\delta\rho(\mathbf{s}) \equiv \rho(\mathbf{s}) - \langle \rho(\mathbf{s}) \rangle$ (for central collisions):

$$\begin{aligned} \varepsilon_2 &= \frac{\delta \mathbf{s}^2}{\langle |\mathbf{s}|^2 \rangle} - \frac{(\delta|\mathbf{s}|^2)(\delta \mathbf{s}^2)}{\langle |\mathbf{s}|^2 \rangle^2} - \frac{(\delta \mathbf{s})^2}{\langle |\mathbf{s}|^2 \rangle} + \cdots \\ \varepsilon_3 &= \frac{\delta \mathbf{s}^3}{\langle |\mathbf{s}|^3 \rangle} - \frac{(\delta|\mathbf{s}|^3)(\delta \mathbf{s}^3)}{\langle |\mathbf{s}|^3 \rangle^2} - 3\frac{(\delta \mathbf{s}^2)(\delta \mathbf{s})}{\langle |\mathbf{s}|^3 \rangle} + \cdots \end{aligned}$$

$$\delta f \equiv \frac{1}{\langle E \rangle} \int_{\mathbf{s}} f(\mathbf{s}) \delta\rho(\mathbf{s}), \langle f \rangle \equiv \frac{1}{\langle E \rangle} \int_{\mathbf{s}} f(\mathbf{s}) \langle \rho(\mathbf{s}) \rangle, \ \langle E \rangle = \int_{\mathbf{s}} \langle \rho(\mathbf{s}) \rangle$$

Cumulants of ε_n

$$
\begin{aligned}
c_n\{2\} &\equiv \langle \varepsilon_n \varepsilon_n^* \rangle \\
c_n\{4\} &\equiv \langle \varepsilon_n \varepsilon_n \varepsilon_n^* \varepsilon_n^* \rangle - 2\langle \varepsilon_n \varepsilon_n^* \rangle \langle \varepsilon_n \varepsilon_n^* \rangle - \langle \varepsilon_n \varepsilon_n \rangle \langle \varepsilon_n^* \varepsilon_n^* \rangle \\
&= \langle \varepsilon_n \varepsilon_n \varepsilon_n^* \varepsilon_n^* \rangle - 2\langle \varepsilon_n \varepsilon_n^* \rangle \langle \varepsilon_n \varepsilon_n^* \rangle \\
SC(3,2) &\equiv \langle \varepsilon_2 \varepsilon_3 \varepsilon_2^* \varepsilon_3^* \rangle - \langle \varepsilon_2 \varepsilon_3 \rangle \langle \varepsilon_2^* \varepsilon_3^* \rangle \\
&\quad - \langle \varepsilon_2 \varepsilon_2^* \rangle \langle \varepsilon_3 \varepsilon_3^* \rangle - \langle \varepsilon_2 \varepsilon_3^* \rangle \langle \varepsilon_3 \varepsilon_2^* \rangle \\
&= \langle \varepsilon_2 \varepsilon_3 \varepsilon_2^* \varepsilon_3^* \rangle - \langle \varepsilon_2 \varepsilon_2^* \rangle \langle \varepsilon_3 \varepsilon_3^* \rangle
\end{aligned}
$$

($SC(3,2)$: Symmetric cumulants)

Their calculation involves the following 2-, 3-, 4-point functions:

2-, 3-, and 4-point functions:

$$C_2(\mathbf{s}_1, \mathbf{s}_2) \equiv \langle \delta\rho(\mathbf{s}_1)\delta\rho(\mathbf{s}_2)\rangle$$

$$C_3(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3) \equiv \langle \delta\rho(\mathbf{s}_1)\delta\rho(\mathbf{s}_2)\delta\rho(\mathbf{s}_3)\rangle$$

$$\begin{aligned} C_4(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4) \equiv\ & \langle \delta\rho(\mathbf{s}_1)\delta\rho(\mathbf{s}_2)\delta\rho(\mathbf{s}_3)\delta\rho(\mathbf{s}_4)\rangle \\ & - \langle \delta\rho(\mathbf{s}_1)\delta\rho(\mathbf{s}_2)\rangle\langle \delta\rho(\mathbf{s}_3)\delta\rho(\mathbf{s}_4)\rangle \\ & - \langle \delta\rho(\mathbf{s}_1)\delta\rho(\mathbf{s}_3)\rangle\langle \delta\rho(\mathbf{s}_2)\delta\rho(\mathbf{s}_4)\rangle \\ & - \langle \delta\rho(\mathbf{s}_1)\delta\rho(\mathbf{s}_4)\rangle\langle \delta\rho(\mathbf{s}_2)\delta\rho(\mathbf{s}_3)\rangle \end{aligned}$$

Special Case of Identical, Independent, Point-Like Sources

$$\rho(\mathbf{s}) = \sum_{j=1}^{N} \delta(\mathbf{s} - \mathbf{s}_j)$$

$$
\begin{aligned}
c_n\{2\} &= \frac{1}{N} \frac{\langle r^{2n} \rangle}{\langle r^n \rangle^2} \\
c_n\{4\} &= \frac{1}{N^3} \left(\frac{\langle r^{4n} \rangle - 2\langle r^{2n} \rangle^2}{\langle r^n \rangle^4} - 8 \frac{\langle r^{3n} \rangle \langle r^{2n} \rangle}{\langle r^n \rangle^5} + 8 \frac{\langle r^{2n} \rangle^3}{\langle r^n \rangle^6} \right) \\
SC(3,2) &= \frac{1}{N^3} \left(\frac{\langle r^{10} \rangle - \langle r^4 \rangle \langle r^6 \rangle}{\langle r^2 \rangle^2 \langle r^3 \rangle^2} - 2 \frac{\langle r^6 \rangle \langle r^7 \rangle}{\langle r^2 \rangle^2 \langle r^3 \rangle^3} - 2 \frac{\langle r^4 \rangle \langle r^8 \rangle}{\langle r^2 \rangle^3 \langle r^3 \rangle^2} \right. \\
&\quad \left. - 6 \frac{\langle r^4 \rangle \langle r^6 \rangle}{\langle r^2 \rangle^2 \langle r^3 \rangle^2} + 4 \frac{\langle r^4 \rangle \langle r^6 \rangle \langle r^5 \rangle}{\langle r^2 \rangle^3 \langle r^3 \rangle^3} + 9 \frac{\langle r^4 \rangle^2}{\langle r^2 \rangle \langle r^3 \rangle^2} \right)
\end{aligned}
$$

N.B. Dependence on the initial density profile is nontrivial.

Standardized Kurtosis

Advantages: (a) Recall $v_n \propto \varepsilon_n (n = 2, 3)$. The constant of proportionality drops out. (b) Meaningful comparison between the above three quantities is possible.

For central collisions:

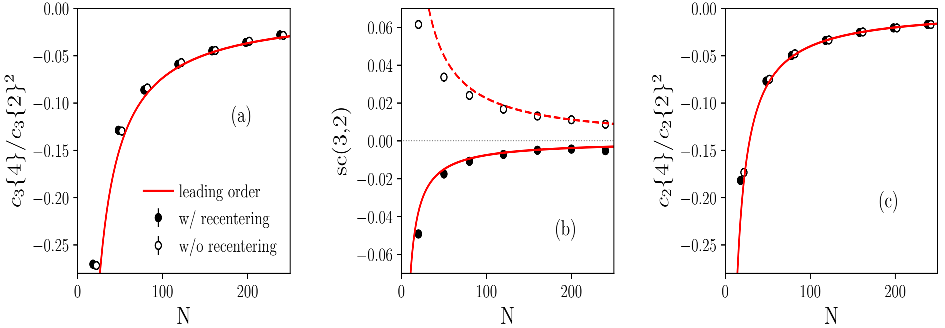
$$\frac{c_2\{4\}}{c_2\{2\}^2} = -\frac{4}{N}$$

$$\frac{c_3\{4\}}{c_3\{2\}^2} = \frac{1}{N}\left(-\frac{69}{2} + \frac{256}{3\pi}\right) \simeq -\frac{7.34}{N}$$

$$sc(3,2) \equiv \frac{SC(3,2)}{c_2\{2\}c_3\{2\}} = -\frac{3}{4N} = -\frac{0.75}{N}$$

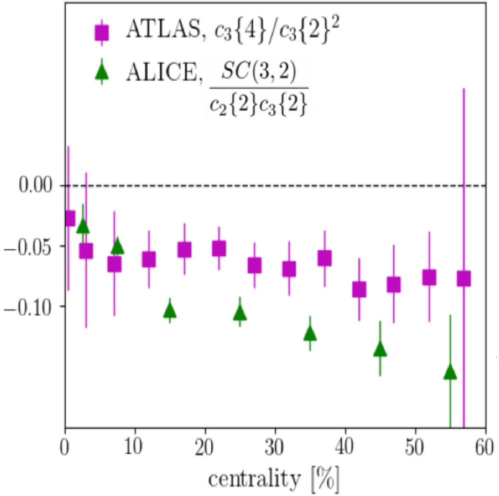
As $N \to \infty$, these vanish, i.e., fluctuations are nearly Gaussian or one recovers the Central Limit Theorem.

Measures of the Initial Kurtosis



(a) Kurtosis of ε_3, (b) Mixed kurtosis of ε_2 and ε_3, (c) Kurtosis of ε_2. Lines: Leading-order perturbative results. Symbols: Monte-Carlo simulations where the average density profile is a symmetric Gaussian. These validate the pert. results.

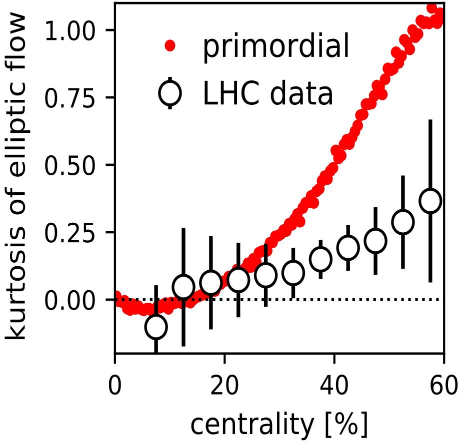
Measures of the Final Kurtosis



- Kurtosis of v_3 fluct, Mixed kurtosis of v_2, v_3
- Small, negative, roughly similar in magnitudes and increasing with centrality percentile
- Same trends as for the **initial** kurtosis

ATLAS 1408.4342, ALICE 1604.07663

Effect of Evolution on the Primordial Non-Gaussianity



Primordial: Trento Model

Data: Pb-Pb, 5.02 TeV
CMS 1711.05594

Evolution partially washes out primordial kurtosis of v_2

Opposite of what happens in cosmology

Primordial $\leftrightarrow$ Initial-state eccentricity ε_2, LHC data $\leftrightarrow$ Final-state elliptic flow v_2. Future high-statistics experiments on v_2 would help constrain $|i\rangle$ models. $\cdots$ 1811.00837 RSB, Giacalone, Ollitrault

Conclusions

- Have presented various measures of non-Gaussian anisotropy fluct: kurtosis of v_3 fluct, mixed kurtosis between v_2 and v_3 (and skewness of v_2 fluct)
- These can be calculated in the hydrodynamic paradigm using $v_n \propto \varepsilon_n \, (n = 2, 3)$ putting new constraints on the $|i>$ models
- Have carried out perturbative expansions of anisotropies in terms of the fluct of $\epsilon_{\rm ini}(x, y)$. Have evaluated kurtosises to leading order, which involves 2-, 3-, 4-point functions of the density field
- The *n*-point functions can, in principle, be calculated from QCD effective theories of $|i>$, linking them to phenomenology. ... Albacete et al. 1808.00795

THANK YOU