The η and η' mesons: spectrum and decay constants

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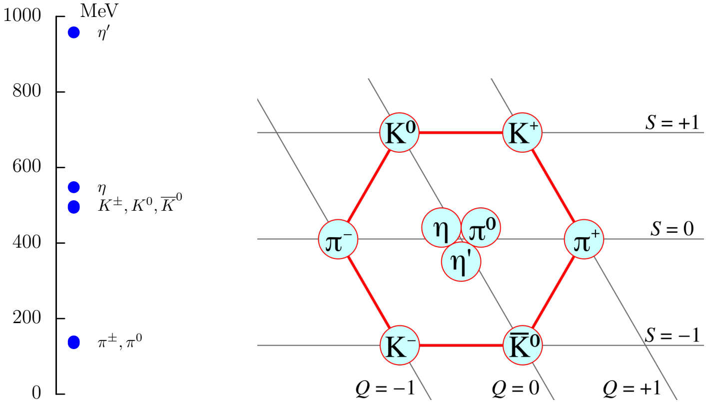
Outline

- ⋆ Properties of η, η' mesons
- ⋆ Simulation details
- ⋆ Extracting η, η' masses and decay constants
- ⋆ Results: continuum limit at physical quark masses, LECs
- ⋆ Singlet axial Ward identity, gluonic decay constants

Pseudoscalar meson nonet



Pseudoscalar meson nonet

If $\exists$ SU(3) flavour symmetry (u,d,s) then for $\bar{q}q$ we have $\bar{3} \otimes 3 = 8 \oplus 1$.

octet: $\pi^0, \pi^\pm, K^\pm, K^0, \overline{K}^0, \eta\,,$ singlet: η'.

$$\eta = \eta_8 \sim \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})\,, \qquad \eta' = \eta_0 \sim \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})\,.$$

Classical global symmetries of $\mathscr{L}_{QCD}$ for $m_u = m_d = m_s$ broken for $m_q \to 0$:

$$\mathrm{SU}_A(3) \times \mathrm{SU}_V(3) \times \mathrm{U}_A(1) \times \mathrm{U}_V(1) \longrightarrow \mathrm{SU}_V(3) \times \mathrm{U}_V(1)$$

$\mathrm{SU}_A(3)$ chiral symmetry spontaneously broken at $T < T_c$,
8 Nambu-Goldstone bosons: $\pi^0, \pi^\pm, K^\pm, K^0, \overline{K}^0, \eta_8$.

$\mathrm{U}_A(1)$ symmetry broken due to quantum corrections (axial anomaly).

η_0 is heavier than octet mesons.

Physical ($m_s > m_u \approx m_d > 0$) η and η' are mixtures of η_8 and η_0.

Axial-Ward Identities

$$\partial_\mu \widehat{A}^{a\mu} = \left(\overline{\psi}\gamma_5\{\widehat{M, t^a}\}\psi\right) + \sqrt{2N_f}\delta^{a,0}\widehat{q}_\mathrm{t}, \qquad a = \mathbf{0}, \ldots, 8\,,$$

where

$$\widehat{A}^{a\mu} = \overline{\psi}\gamma_\mu\gamma_5 t^a\psi, \qquad M = \mathrm{diag}\,(m_u, m_d, m_s)$$

with the topological charge density

$$q_\mathrm{t}(x) = -\frac{1}{16\pi^2}\,\mathrm{tr}\left[F_{\mu\nu}(x)\widetilde{F}_{\mu\nu}(x)\right], \qquad Q_\mathrm{t} = \int \mathrm{d}^4x\, q_\mathrm{t}(x)\,.$$

Witten and Veneziano relation (large N_c limit):

$$\frac{F_\pi^2}{2N_f}\left(M_\eta^2 + M_{\eta'}^2 - 2M_K^2\right) = \chi_\mathrm{top}, \qquad \chi_\mathrm{top} = \frac{\langle\widehat{Q}_\mathrm{t}^2\rangle}{V},$$

where χ_top is the quenched chiral susceptibility.

Axial decay constants

Local axial-vector currents:

$$\langle 0|\widehat{A}^{a\mu}|\mathcal{M}\rangle = i\,F^{a}_{\mathcal{M}}p^{\mu}, \qquad \mathcal{M} = \eta, \eta'$$

(normalized so that physical $F_\pi = F^3_{\pi^0} \approx 92$ MeV)

Four independent decay constants:

$$\begin{pmatrix} F^8_\eta & F^0_\eta \\ F^8_{\eta'} & F^0_{\eta'} \end{pmatrix} = \begin{pmatrix} F^8\cos\theta_8 & -F^0\sin\theta_0 \\ F^8\sin\theta_8 & F^0\cos\theta_0 \end{pmatrix} = \begin{pmatrix} \cos\theta_8 & -\sin\theta_0 \\ \sin\theta_8 & \cos\theta_0 \end{pmatrix} \begin{pmatrix} F^8 & 0 \\ 0 & F^0 \end{pmatrix}$$
$$= \Xi(\theta_8, \theta_0)\,\mathrm{diag}(F^8, F^0)\,.$$

In the SU(3) limit ($m_u = m_d = m_s$): $\theta_8 = \theta_0 = 0$.

One may also use the "flavour basis": $\bar{\ell}\ell = \bar{u}u = \bar{d}d$ and $\bar{s}s$:

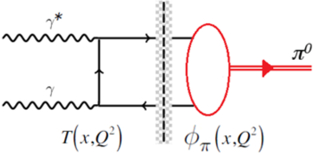
$$\begin{pmatrix} F^\ell_\eta & F^s_\eta \\ F^\ell_{\eta'} & F^s_{\eta'} \end{pmatrix} = \Xi(\theta_\ell, \theta_s)\mathrm{diag}(F^\ell, F^s) = \frac{1}{\sqrt{3}} \begin{pmatrix} F^8_\eta + \sqrt{2}F^0_\eta & -\sqrt{2}F^8_\eta + F^0_\eta \\ \sqrt{2}F^0_{\eta'} + F^8_{\eta'} & F^0_{\eta'} - \sqrt{2}F^8_{\eta'} \end{pmatrix}.$$

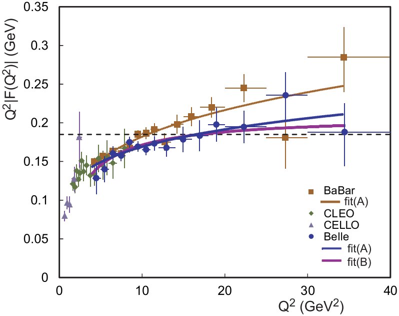
Distribution amplitudes of η, η'

Nonperturbative input for the theoretical description of hard exclusive processes.

Collinear factorization at large Q^2. E.g., pion: $\gamma\gamma^* \to \pi$ form factor.

[Belle,1205.3249]





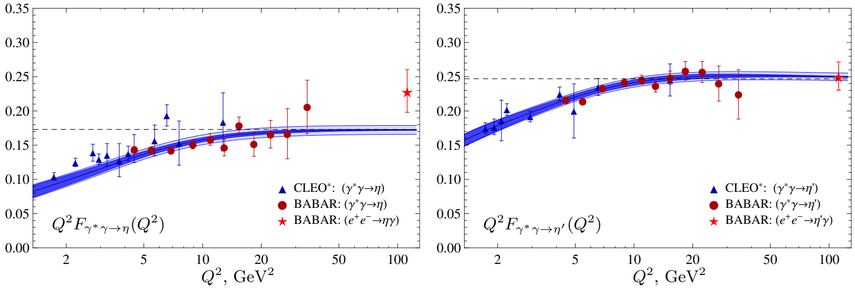
$$\mathbf{F}_{\pi\gamma}(\mathbf{Q^2}) = \frac{2\sqrt{2}F_\pi}{3}\int_0^1 \mathrm{d}x\, \underbrace{T_H(x,\mu,Q^2)}_{\text{hard}}\,\underbrace{\phi_\pi(x,\mu)}_{\text{soft}}\,.$$

$\gamma\gamma^\star \to \eta/\eta'$ form factor ($\mathcal{M} \in \{\eta,\eta'\}$):

$$\mathbf{F}_{\gamma^\star\gamma\to\mathcal{M}}(\mathbf{Q^2}) = \frac{\sqrt{2}F^8_{\mathcal{M}}}{3\sqrt{6}}\int_0^1 \mathrm{d}x\, T^8_H(x,\mu,Q^2)\phi^8_{\mathcal{M}}(x,\mu) + \frac{2\sqrt{2}F^0_{\mathcal{M}}}{3\sqrt{3}}\int_0^1 \mathrm{d}x\, T^0_H(x,\mu,Q^2)\phi^0_{\mathcal{M}}(x,\mu) + \frac{\sqrt{2}F^0_{\mathcal{M}}}{3\sqrt{3}}\int_0^1 \mathrm{d}x\, T^g_H(x,\mu,Q^2)\phi^g_{\mathcal{M}}(x,\mu)\,.$$

$\gamma\gamma^\star \to \eta/\eta'$ form factors

[Agaev,1409.4311]



Expt: [BABAR,1101.1142], [CLEO,hep-ex/9707031], not shown [BABAR,1808.08038].

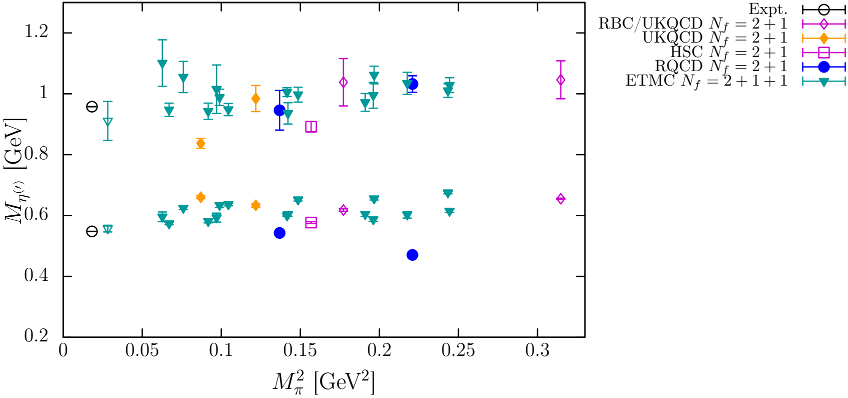
Gegenbauer expansion: $\phi^{0,8}_{\mathcal{M}} = 6x(1-x)\left[1 + \sum_{n=2,4,\ldots} c^{0,8}_{n,\mathcal{M}} C^{3/2}_n(2x-1)\right]$,

$\phi^{g}_{\mathcal{M}} = 30x^2(1-x)^2 \sum_{n=2,4,\ldots} c^{g}_{n,\mathcal{M}} C^{5/2}_{n-1}(2x-1)$

$$\lim_{Q^2\to\infty} Q^2 F_{\gamma^*\gamma\to\mathcal{M}}(Q^2) = \frac{2}{\sqrt{3}}\left[\mathbf{F}^{\mathbf{8}}_{\mathcal{M}} + 2\sqrt{2}\,\mathbf{F}^{\mathbf{0}}_{\mathcal{M}}(\mu_0)\left(1 - \frac{2N_f}{\pi\beta_0}\alpha_s(\mu_0)\right)\right].$$

Relevant to model part of the hadronic light-by-light contribution to a_μ, see e.g.: [Escribano,1307.2061], [Jegerlehner,1511.04473], [Colangelo,1910.11881].

Previous lattice work: masses



[ETMC,1710.07986]: continuum, chiral extrapolation (▽).

$M_\eta = 577(11)_{\text{stat}}(03)_{\text{ChPT}}$ MeV $\qquad M_{\eta'} = 911(64)_{\text{stat}}(03)_{\text{ChPT}}$ MeV

Previous lattice work: decay constants

[ETMC,1710.07986]: use the flavour basis and ChPT with Feldmann-Kroll-Stech scheme [hep-ph/9802409].

F^0 and hence F^ℓ and F^s are assumed to be scale independent.

Indirect determination via pseudoscalar amplitudes [Feldmann,hep-ph/9907491].

$$h^a_{\mathcal{M}} = 2m_a\langle 0|P^a|\mathcal{M}\rangle, \qquad a \in \{\ell, s\}\,.$$

In leading order ChPT

$$\begin{pmatrix} h^\ell_\eta & h^s_\eta \\ h^\ell_{\eta'} & h^s_{\eta'} \end{pmatrix} = \Xi(\theta,\theta)\mathrm{diag}\left(M_\pi^2 f_\ell, (2M_K^2 - M_\pi^2) f_s\right)\,.$$

Expected $|\theta_\ell - \theta_s|/|\theta_\ell + \theta_s| \ll 1 \Rightarrow \theta_\ell, \theta_s \mapsto \theta$.

Find after continuum and chiral extrapolation:

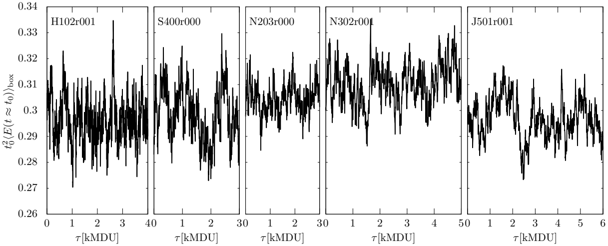
$$f^\ell = \sqrt{2}F^\ell = 125(5)_{\mathrm{stat}}(6)_{\mathrm{ChPT}}\ \mathrm{MeV}\,, \qquad f^s = \sqrt{2}F^s = 178(4)_{\mathrm{stat}}(1)_{\mathrm{ChPT}}\ \mathrm{MeV}\,.$$

$N_f = 2 + 1$ CLS ensembles

Coordinated Lattice Simulations (CLS): Berlin, CERN, Mainz, UA Madrid, Milano Bicocca, Münster, Odense, Regensburg, Rome I and II, Wuppertal, DESY-Zeuthen.

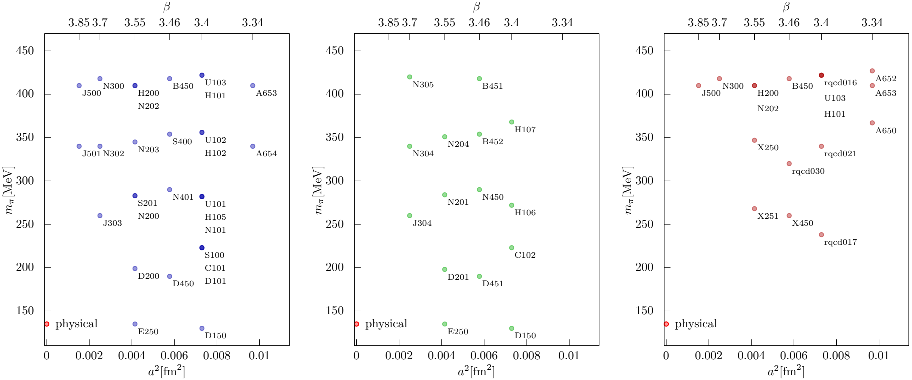
- ⋆ Non-perturbatively improved clover fermion action and tree-level Lüscher-Weisz gauge action.
- ⋆ Six (five) lattice spacings: $a = 0.1 - 0.04$ fm.
- ⋆ $LM_\pi \gtrsim 4$ and multiple spatial volumes.
- ⋆ Mostly open boundary conditions in time.

Wilson flow action density, $t_0^2 E(t \approx t_0)$, $M_\pi \approx 340$ MeV, averaged over ≈ 1 fm slice.



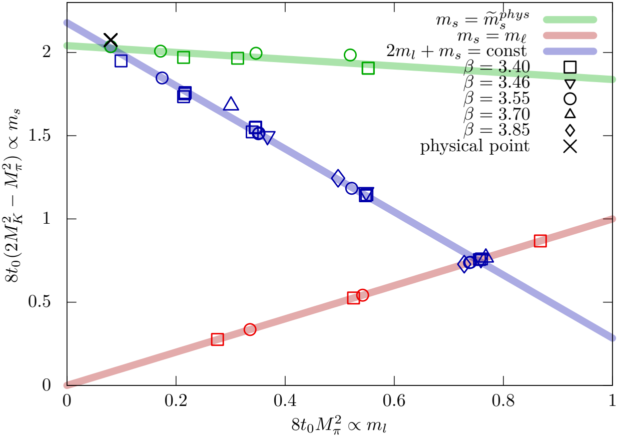
CLS ensembles: M_π vs a^2

- ⋆ Three trajectories, physical point ensembles.
- ⋆ Typically 6000–10000 MDUs.



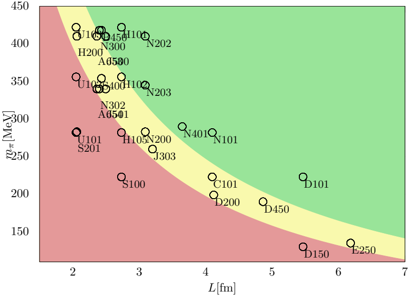
$2m_\ell + m_s = \text{const.}$ $\quad$ $m_s = \text{const.}$ $\quad$ $m_\ell = m_s$

CLS ensembles: m_ℓ-m_s plane

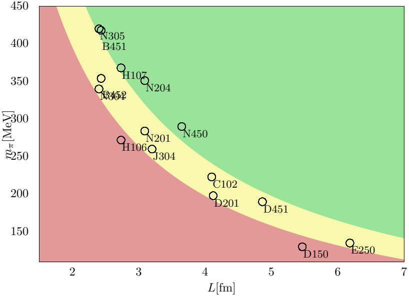


CLS ensembles: spatial volume

$2m_\ell + m_s = \mathrm{const.}$



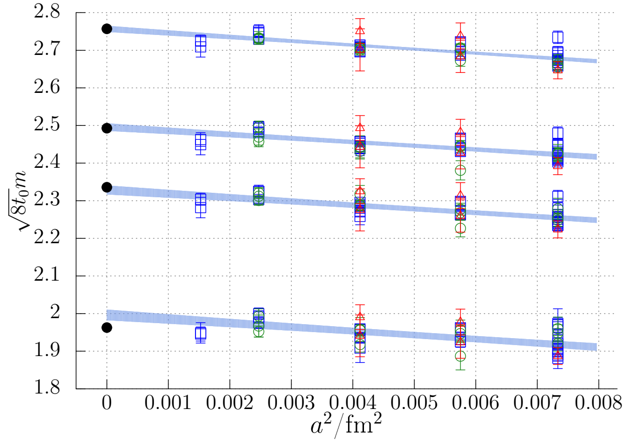
$m_s = \mathrm{const.}$



$\mathbf{LM_\pi < 4}$ $\quad$ $\mathbf{4 \leq LM_\pi < 5}$ $\quad$ $\mathbf{LM_\pi \geq 5}$

Scale setting, the continuum limit

Ξ, Σ, Λ, N. Data shifted to physical quark mass using SU(3) BChPT fit.



a^2 varied by a factor of 4.6.

Compatible with $\sqrt{8t_{0,\mathrm{ph}}} = 0.413(6)$ fm from $F_\pi + 2F_K$ [ALPHA,1608.08900].

η and η' on the lattice

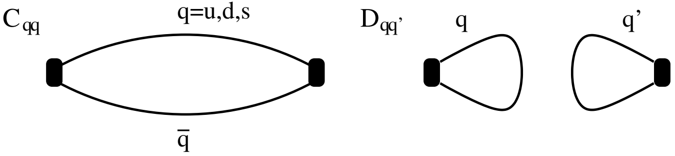
Choose a basis of operators: e.g.

$$b_8 = \frac{1}{\sqrt{6}}(u\gamma_5\bar{u} + d\gamma_5\bar{d} - 2s\gamma_5\bar{s}), \qquad b_0 = \frac{1}{\sqrt{3}}(u\gamma_5\bar{u} + d\gamma_5\bar{d} + s\gamma_5\bar{s})$$

Construct a matrix of correlators:

$$C_{ij}(t) = \frac{1}{N_t}\sum_{t_i=0}^{N_t-1}\langle b_i(t+t_i)b_j^\dagger(t_i)\rangle$$

Evaluate quark line diagrams: e.g.



$$C_{88} = \frac{1}{3}\left(C_{\ell\ell} + C_{ss} - 2D_{\ell\ell} - 2D_{ss} + 2D_{\ell s} + 2D_{s\ell}\right)$$

Extract masses by simultaneously fitting to

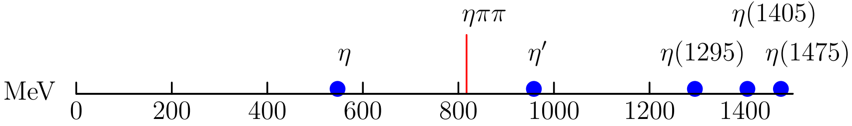
$$C_{ij}(t) \longrightarrow \langle 0|b_i|\eta\rangle\langle\eta|b_j^\dagger|0\rangle \frac{e^{-E_\eta t}}{2E_\eta V_3} + \langle 0|b_i|\eta'\rangle\langle\eta'|b_j^\dagger|0\rangle \frac{e^{-E_{\eta'} t}}{2E_{\eta'} V_3} + \dots$$

Signal for C_{ij} dies away quickly in time (D_{qq} typically noisy).

$$C_{qq} = C_{qq}(t) \sim e^{-E_\pi t}, \qquad D_{qq}(t) \sim e^{-E_{\eta/\eta'} t} - e^{-E_\pi t}$$

Forced to extract masses at early times.

However, operators create all states with the same QNs (with varying overlap).



Need to take nearby excited states into account.

Use spatially extended ("smeared") operators to improve the overlap with the ground state.

Strong decay $\eta' \to \eta\pi\pi$ is possible in our set-up but the experimental width is small: $\Gamma \sim 130$ keV. We neglect this.

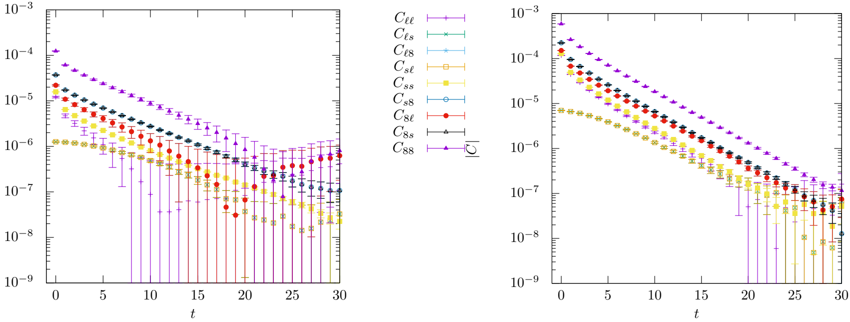
Correlator signal

$m_\ell \neq m_s$: basis of three operators (b_ℓ, b_s, b_8, with different smearings).

$m_\ell = m_s$: basis of four operators (b_8, b_0, two smearings) with C_{ij} being block diagonal.

Fit both $|\vec{p}| = 0$ and $|\vec{p}| = 2\pi/L$ correlators including a term for the first excited state.

Ensemble D200: $M_\pi \sim 200$ MeV, $a = 0.064$ fm. $C_{\ell\ell} = \frac{1}{N_t}\sum_{t_i=0}^{N_t-1}\langle b_\ell(t+t_i) b_\ell^\dagger(t_i)\rangle$ etc..

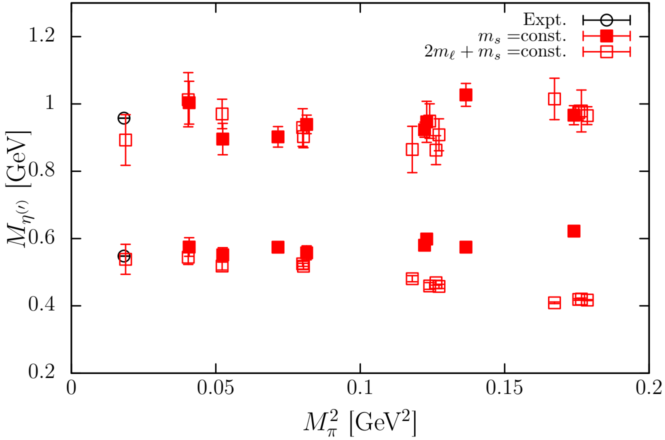


$|\vec{p}| = 0$

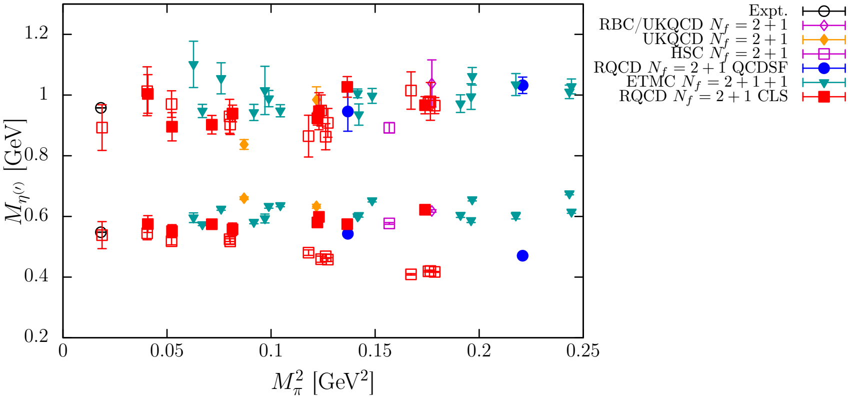
$|\vec{p}| = 2\pi/L$

Mass spectrum

- 22 ensembles.
- $M_\pi = 420 - 135$ MeV
- $a = 0.086,\ 0.076,\ 0.064$ fm (and 0.050 fm).
- $M_\pi L \gtrsim 4$



Mass spectrum: comparison with other results



Decay constants

Extracted from fits to

$$C_i^{A^{a\mu}}(t) = \langle A^{a\mu}(t) b_i^\dagger(0)\rangle, \qquad a \in \{0,8\}\,,$$

where

$$C_i^{A^{a\mu}}(t) \longrightarrow \frac{e^{-E_\eta t}}{2E_\eta V_3}\langle 0|A^{a\mu}|\eta\rangle\langle\eta|b_i|0\rangle + \frac{e^{-E_{\eta'} t}}{2E_{\eta'} V_3}\langle 0|A^{a\mu}|\eta'\rangle\langle\eta'|b_i|0\rangle + \ldots$$

Energies and overlaps $\langle\mathcal{M}|b_i|0\rangle$ fixed from $C_{ij}(t)$ fits.

Renormalization and improvement: [Bhattacharya,hep-lat/0511014]

$$\widehat{A}^{8\mu} = Z_A^{ns}\left(1 + b_A am^8 + 3\tilde{b}_A am^0\right) A^{a\mu}_{\rm imp}, \qquad \widehat{A}^{0\mu} = Z_A^{s}\left([1 + 3\tilde{d}_A am^0] A^{0\mu}_{\rm imp} + a d_A^0 \mathrm{Tr}[MA^\mu]\right)$$

$$A^{8\mu}_{\rm imp} = A^{8\mu} + a c_A^{ns}\partial_\mu P^8, \qquad A^{0\mu}_{\rm imp} = A^{a\mu} + a c_A^{s}\partial_\mu P^0,$$

$$m^8 = \frac{1}{3}(m_\ell + 2m_s)\,, \qquad m^0 = \frac{1}{3}(2m_\ell + m_s)\,, \qquad am_q = \frac{1}{2}\left(\frac{1}{\kappa_q} - \frac{1}{\kappa_{\rm crit}}\right)\,.$$

Z_A^{ns} [Brida,1808.09236], c_A^{ns} [ALPHA,1502.04999] and b_A [Korcyl,1607.07090] known non-perturbatively.

Use perturbative result for $Z_A^s - Z_A^{ns}$ [Constantinou,1610.06744] ($O(\alpha_s^2)$). Also impose $c_A^s = c_A^{ns} \rightarrow$ residual $O(a)$ effects.

Renormalization of the singlet axial current

The singlet axial currents depend on the QCD scale μ [Constantinou,1610.06744]:

$$Z_A^s(g^2,\mu) - Z_A^{ns}(g^2) = -a_s^2(a^{-1})\left[\frac{3}{2}\log(a^2\mu^2) - 2.834(11)\right] + \ldots\,,$$

where $a_s = \alpha_s/\pi$. Evolution with μ depends on $\beta(a_s)$ and $\gamma_A(a_s)$.

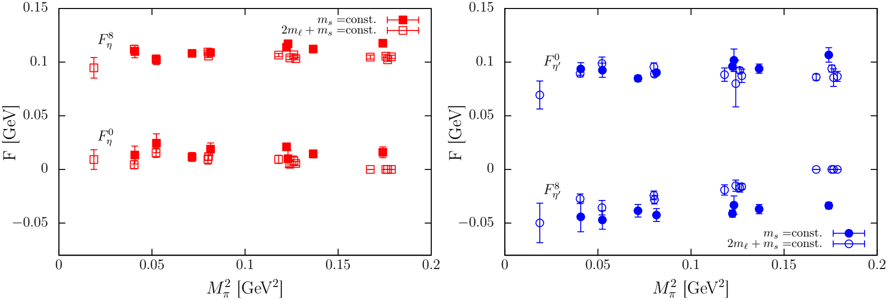
Since $\gamma_A^s(a_s) = -\gamma_{A1}^s a_s^2 - \gamma_{A2}^s a_s^3 - \ldots$, i.e. $\gamma_{A0}^s = 0$, $Z_A^s(\mu=\infty) = Z_A^{s\prime}$ is finite.

Use Z_A^s in the $\overline{\text{MS}}$ scheme at $\mu = a^{-1}$ and evolve to $\mu = \infty$, defining a scale-independent renormalization [Zöller,1304.2232]:

$$Z_A^{s\prime}(g^2) = \exp\left\{-\left[\frac{\gamma_{A1}^s}{\beta_0}a_s(\mu) + \frac{1}{2}\left(\frac{\gamma_{A2}^s}{\gamma_{A1}^s} - \frac{\beta_1}{\beta_0}\right)a_s^2(\mu)\right]\right\}\cdot Z_A^s(g^2,\mu)\,.$$

Final results can be converted to low scales, using the ratio of the above conversion factors $Z_A^s(\mu)/Z_A^{s\prime}$.

Decay constants in the $\overline{\mathrm{MS}}'$ scheme



For ensembles with $m_\ell = m_s$: $F_\eta^0 = F_{\eta'}^8 = 0$.

Also employ $|\vec{p}| = 0$ and $|\vec{p}| = 2\pi/L$ correlators to extract $F_{\eta,\eta'}^{0,8}$.

Quark mass and continuum limit extrapolation

Perform simultaneous $a \to 0$ and $m_q \to m_q^{\rm phys}$ extrapolation.
$(\delta M^2 = 2(M_K^2 - M_\pi^2),\ \overline{M}^2 = (2M_K^2 + M_\pi^2)/3)$:

$$\mathbb{M}_{\mathcal{M}}(\mathbb{M}_\pi, \mathbb{M}_K, \mathbb{a}) = \mathbb{m}_{\mathcal{M}}(\mathbb{M}_\pi, \mathbb{M}_{\mathbf{K}}, \mathbf{0}) \left[1 + c_{\mathcal{M}}\mathbb{a}^{\mathbf{2}} + \bar{c}_{\mathcal{M}}\mathbb{a}^{\mathbf{2}}\overline{\mathbb{M}}^2 + \delta c_{\mathcal{M}}\mathbb{a}^{\mathbf{2}}\delta\mathbb{M}^2\right],$$

$$\mathbb{M}_{\mathcal{M}} = \sqrt{8t_0}M_{\mathcal{M}}.$$

It is convenient in ChPT to use singlet/octet basis:

$$\begin{pmatrix} F_\eta^8 & F_\eta^0 \\ F_{\eta'}^8 & F_{\eta'}^0 \end{pmatrix} = \begin{pmatrix} F^8\cos\theta_8 & -F^0\sin\theta_0 \\ F^8\sin\theta_8 & F^0\cos\theta_0 \end{pmatrix} = \Xi(\theta_8, \theta_0)\,\mathrm{diag}(F^8, F^0)\,.$$

$$\mathbb{F}^8(\mathbb{M}_\pi, \mathbb{M}_K, \mathbb{a}) = \mathbb{F}^{\mathbf{8}}(\mathbb{M}_\pi, \mathbb{M}_{\mathbf{K}}, \mathbf{0}) \left[1 + \bar{d}_8'\mathbb{a}\overline{\mathbb{M}}^2 + d_8\mathbb{a}^2 + \bar{d}_8\mathbb{a}^2\overline{\mathbb{M}}^2 + \delta d_8\mathbb{a}^2\delta\mathbb{M}^2\right]$$

$$\mathbb{F}^0(\mathbb{M}_\pi, \mathbb{M}_K, \mathbb{a}) = \mathbb{F}^{\mathbf{0}}(\mathbb{M}_\pi, \mathbb{M}_{\mathbf{K}}, \mathbf{0}) \left[1 + d_0\mathbb{a} + \bar{d}_0\mathbb{a}\overline{\mathbb{M}}^2 + \delta d_0\mathbb{a}\delta\mathbb{M}^2\right]$$

$$\mathbb{F}^{8,0} = \sqrt{8t_0}F^{8,0} = \sqrt{8t_0\left({F_\eta^{8,0}}^2 + {F_{\eta'}^{8,0}}^2\right)}.$$

Similarly $\theta_8 = \arctan(F_{\eta'}^8/F_\eta^8)$ ($O(a^2)$), $\theta_0 = \arctan(-F_\eta^0/F_{\eta'}^0)$ ($O(a)$ and $O(a^2)$).

Large-N_c ChPT

U(3) EFT, η' becomes a pseudo-Goldstone boson in the t'Hooft limit.
Expansion: $p^2 = O(\delta)$, $m = O(\delta)$, $1/N_c = O(\delta)$.

Known up to NNLO, e.g. [Guo,1503.02248] [Bickert,1612.05473], use NLO.
Mass matrix in the effective Lagrangian:

$$\mathscr{L}_{\text{eff}}^{M} = -\frac{1}{2}\eta_A^T \mathcal{M}_A^2 \eta_A, \qquad \eta_A = \begin{pmatrix} \eta_8 \\ \eta_0 \end{pmatrix} \qquad \mathcal{M}_A^2 = \begin{pmatrix} \mu_{\mathbf{8}} & \mu_{\mathbf{80}} \\ \mu_{\mathbf{80}} & \mu_{\mathbf{0}} \end{pmatrix}$$

with

$$\mathcal{M}_A^2 = R^T \begin{pmatrix} M_\eta^2 & 0 \\ 0 & M_{\eta'}^2 \end{pmatrix} R, \qquad R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$M_{\eta'}^2 + M_\eta^2 = \mu_{\mathbf{8}} + \mu_{\mathbf{0}}, \qquad M_{\eta'}^2 - M_\eta^2 = \sqrt{(\mu_{\mathbf{8}} - \mu_{\mathbf{0}})^2 + 4{\mu_{\mathbf{80}}}^2},$$

$$\theta = \frac{1}{2}\arcsin\left(2\mu_{\mathbf{80}}/\sqrt{(\mu_{\mathbf{8}} - \mu_{\mathbf{0}})^2 + 4{\mu_{\mathbf{80}}}^2}\right)$$

$$\mu_{\mathbf{8}} = (4M_K^2 - M_\pi^2)/3 + \text{NLO}, \qquad \mu_{\mathbf{0}} = (2M_K^2 + M_\pi^2)/3 + {\mathbf{M_0}}^2 + \text{NLO},$$

$$\mu_{\mathbf{80}} = -\frac{2\sqrt{2}}{3}\left(M_K^2 - M_\pi^2\right) + \text{NLO}, \qquad {\mathbf{M_0}}^2 = 2N_f\chi_{\text{top}}/F_\pi^2$$

Large-N_c ChPT

Decay constants:

$$F^8 = F_0\left(1+\frac{\delta_8^{(1)}}{2}\right) \qquad \delta_8^{(1)} = \frac{8(4M_K^2-M_\pi^2)\mathbf{L_5}}{3F_\pi^2}$$

$$F^0 = F_0\left(1+\frac{\delta_0^{(1)}}{2}\right) \qquad \delta_0^{(1)} = \frac{8(2M_K^2+M_\pi^2)\mathbf{L_5}}{3F_\pi^2} + \Lambda_1$$

$$\theta_8 = \theta^{[1]} + \arctan\left(\frac{\delta_{80}^{(1)}}{2}\right) \qquad \theta_0 = \theta^{[1]} - \arctan\left(\frac{\delta_{80}^{(1)}}{2}\right) \qquad \delta_{80}^{(1)} = -\frac{16(M_K^2-M_\pi^2)\mathbf{L_5}}{3F_\pi^2}$$

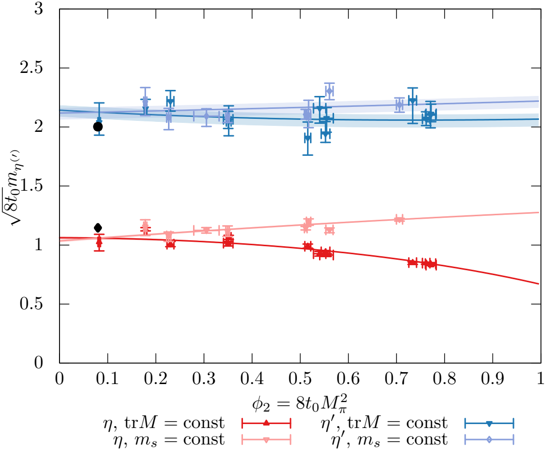
Low energy constants:

- LO: M_0^2, F_0 (F_0: pion decay constant in the $N_f = 3$ chiral limit).
- NLO: L_5, L_8, Λ_1, Λ_2,
- NNLO (11 additional LECs): L_4, L_7, ...

Appear in expressions for both masses and decay constants

$\rightarrow$ perform a joint fit.

Quark mass and continuum extrapolation: LO ChPT



- ⋆ To this order $F^{8,0}$ are constant (but data are not) $\to$ only fit the masses.
- ⋆ χ^2/d.o.f. $= 2.6$, M_η at $a = 0$, M_π^{phys} significantly below experiment. (also when performing cuts in $\overline{M}^2 = (2M_k^2 + M_\pi^2)/3$)
- ⋆ Discretization effects are removed using the fit. Included: δc_η, $c_{\eta'}$, $\overline{c}_{\eta'}$

Mass and continuum extrapolation: Gasser and Leutwyler

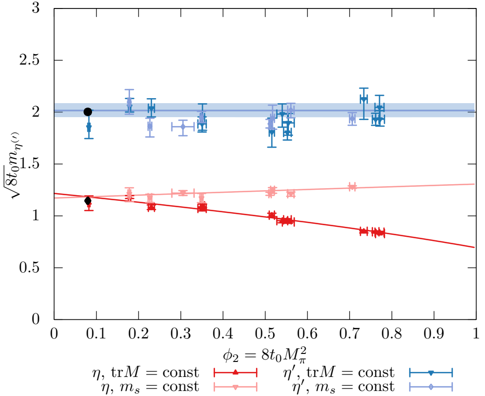
$$M^2_{\eta'} + M^2_{\eta} = M^2_0 + \tilde{M}^2_{\eta_8} \qquad M^2_{\eta'} - M^2_{\eta} = \frac{M^2_0 - \tilde{M}^2_{\eta_8}}{\cos 2\delta},$$

$$\tilde{M}^2_{\eta_8} = \frac{1}{3}(4M^2_K - M^2_\pi), \qquad \tan 2\delta = -\frac{4}{3}\sqrt{2}\gamma \frac{M^2_K - M^2_\pi}{M^2_0 - \tilde{M}^2_{\eta_8}}$$

[Gasser,Nucl. Phys. B250 (1985)]
LO Large-N_c ChPT: $\gamma = 1/2$

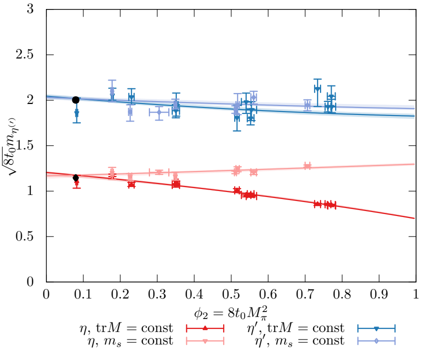
$\chi^2/\text{d.o.f.} = 1.3$

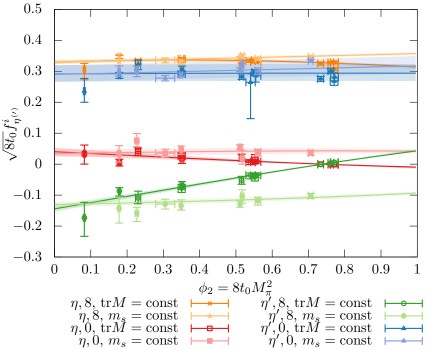
a effects: Only $c_{\eta,\eta'}$.



Quark mass and continuum extrapolation: NLO ChPT

Simultaneous fit to the masses and decay constants.





χ^2/d.o.f. $= 1.01$, NLO ChPT ($+a$ effects) provides a reasonable description of the data.

a effects: all but two.

Results

$$
\begin{aligned}
M_\eta &= 561.6\,\binom{3.8}{11.5}_{\text{stat}}\,\binom{1.6}{1.9}_{a}\,\binom{16.0}{0}_{\text{ChPT}}\,(8.2)_{t_0},\ \text{MeV} && {}^{+3\%}_{-3\%}\\
M_{\eta'} &= 964.3\,\binom{9.6}{9.1}_{\text{stat}}\,\binom{5}{3.3}_{a}\,\binom{4.9}{6.2}_{\text{ChPT}}\,(14.0)_{t_0}\ \text{MeV} && {}^{+2\%}_{-2\%}\\
F^8_\eta &= 112.2\,\binom{1.6}{1.7}_{\text{stat}}\,\binom{4}{6.1}_{a}\,\binom{4}{0}_{\text{ChPT}}\,(1.6)_{t_0}\ \text{MeV} && {}^{+6\%}_{-7\%}\\
F^8_{\eta'} &= -43.4\,\binom{2.7}{2.5}_{\text{stat}}\,\binom{2.1}{1.6}_{a}\,\binom{0}{5.9}_{\text{ChPT}}\,(6)_{t_0}\ \text{MeV} &&\\
F^0_\eta(\mu=\infty) &= 12.0\,\binom{3.6}{3.6}_{\text{stat}}\,\binom{6.1}{1.5}_{a}\,\binom{0}{2.2}_{\text{ChPT}}\,(2)_{t_0}\,\binom{0}{1.0}_{\text{ren}}\ \text{MeV} &&\\
F^0_{\eta'}(\mu=\infty) &= 98.56\,\binom{9.59}{8.01}_{\text{stat}}\,\binom{2.51}{97}_{a}\,\binom{9.98}{0}_{\text{ChPT}}\,(1.43)_{t_0}\,\binom{0}{3.18}_{\text{ren}}\ \text{MeV} && {}^{+14\%}_{-9\%}
\end{aligned}
$$

Lattice spacing set using $\sqrt{8t_0} = 0.413(6)$ fm [ALPHA,1608.08900].

Renormalization error for $F^0_{\eta,\eta'}$ from employing $\mu = a^{-1}/2$ and $2a^{-1}$. Central value from $\mu = a^{-1}$.

Results

Flavour basis ($\mu = \infty$):

$$F^\ell = 92.92\,\binom{5.53}{5.68}_{\text{stat}}\binom{2.41}{4.83}_a\binom{3.15}{0}_{\text{ChPT}}(1.35)_{t_0}\binom{0}{2.06}_{\text{ren}}\ \text{MeV}$$
$$F^s = 125.3\,\binom{3.1}{2.6}_{\text{stat}}\binom{0}{4.0}_a\binom{8.9}{0}_{\text{ChPT}}(1.8)_{t_0}\binom{0}{1.1}_{\text{ren}}\ \text{MeV}$$
$$\theta_\ell = 0.639\,\binom{47}{45}_{\text{stat}}\binom{14}{15}_a\binom{52}{0}_{\text{ChPT}}\binom{0}{16}_{\text{ren}},$$
$$\theta_s = 0.742\,\binom{50}{47}_{\text{stat}}\binom{25}{56}_a\binom{0}{46}_{\text{ChPT}}\binom{15}{0}_{\text{ren}}$$

[ETMC,1710.07986] **indirect determination, μ unknown.**

$$F^\ell = 88(4)_{\text{stat}}(4)_{\text{ChPT}}\ \text{MeV}$$
$$F^s = 126(3)_{\text{stat}}(1)_{\text{ChPT}}\ \text{MeV}$$
$$\theta = 38.8(2.2)_{\text{stat}}(2.4)^\circ_{\text{ChPT}} = 0.677(38)_{\text{stat}}(42)_{\text{ChPT}}$$

$\gamma\gamma^\star \to \eta/\eta'$ form factors at large Q^2:

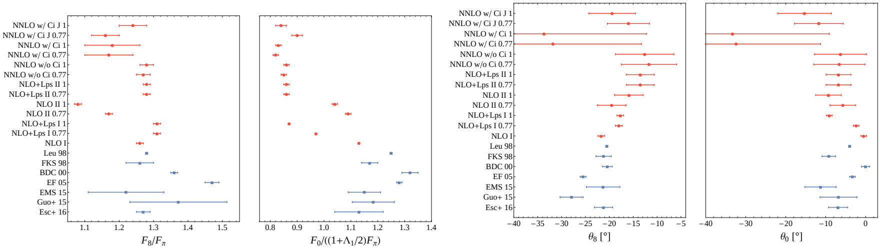
$$Q^2 F_{\gamma\gamma^\star\to\eta}(Q^2) = 155(15)_{\text{stat}}(23)_{\text{ChPT}}\ \text{MeV}$$
$$Q^2 F_{\gamma\gamma^\star\to\eta'}(Q^2) = 277(09)_{\text{stat}}(01)_{\text{ChPT}}\ \text{MeV}$$

Our analysis (glue does not contribute at leading twist):

$$Q^2 F_{\gamma\gamma^\star\to\eta}(Q^2) = 168(11)_{\text{stat}}\binom{18}{9}_a\binom{0}{7}_{\text{ChPT}}(2)_{t_0}\ \text{MeV}$$
$$Q^2 F_{\gamma\gamma^\star\to\eta'}(Q^2) = 272\,\binom{30}{24}_{\text{stat}}\binom{8}{4}_a\binom{0}{26}_{\text{ChPT}}(4)_{t_0}\ \text{MeV}$$

Phenomenological determinations

[Bickert,1612.05473] Note that the scale dependence is typically ignored!

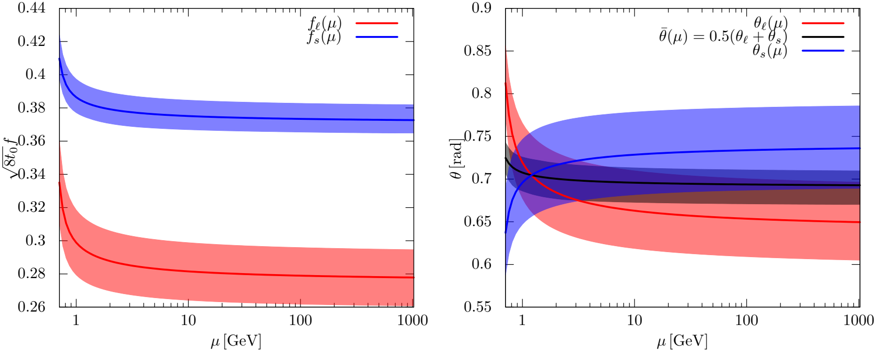


Statistical errors only:

$$F^8/F_\pi = 1.309^{+14}_{-13}, \quad F^0/([1+\Lambda_1/2]F_\pi) = 1.22^{+10}_{-9},$$

$$\theta_8 = -21^{\circ}{}^{+1}_{-1}, \quad \theta_0 = -7^{\circ}{}^{+2}_{-1}.$$

Scale dependence of $F^{\ell,s}$ and $\theta_{\ell,s}$



NLO Large-N_c ChPT low energy constants

Determined **fixing M_η and $M_{\eta'}$ to expt.**.

$$L_5 = 0.00158\,\binom{9}{8}_{\rm stat}\binom{0}{18}_a\binom{37}{0}_{\rm ChPT}\,,\qquad L_8 = 9.11\,\binom{85}{1.04}_{\rm stat}\binom{1.07}{12}_a\binom{1.83}{0}_{\rm ChPT}\times 10^{-4},$$
$$M_0 = 801\,\binom{21}{19}_{\rm stat}\binom{73}{0}_a\binom{9}{0}_{\rm ChPT}(12)_{t_0}\binom{1}{4}_{\rm ren}\ {\rm MeV},\qquad F_0 = 99.9\,\binom{1.8}{2.0}_{\rm stat}\binom{1.2}{2.5}_a\binom{0}{4.0}_{\rm ChPT}(1.5)_{t_0}\ {\rm MeV}$$
$$\Lambda_1 = -0.30\,\binom{8}{6}_{\rm stat}\binom{25}{0}_a\binom{3}{0}_{\rm ChPT}\binom{0}{2}_{\rm ren}\,,\qquad \Lambda_2 = -0.070\,\binom{89}{42}_{\rm stat}\binom{209}{0}_a\binom{143}{0}_{\rm ChPT}\binom{0}{11}_{\rm ren}\,.$$

[Guo,1803.07284]: fit $M_{\eta,\eta'}$, $F_{\eta,\eta'}$ from [ETMC,1710.07986] (corrected by ETMC fit for m_s and a), M_K, F_π, F_K from [RBC/UKQCD,1208.4412] and F_K/F_π from [BMW,1001.4692] and others for $M_{\eta,\eta'}$.

- LO: $M_0 \sim 820$ MeV (fit to masses)
- NLO: $F_0 \sim 91$ MeV, $L_5 \sim 0.002$, $L_8 \sim 9\times 10^{-4}$, $\Lambda_1 \sim -0.2$, $\Lambda_2 \sim 0.02$
- NNLO: $F_0 = 81 - 92$ MeV, ...

[Hernández,1907.11511]: $F^{N_f=3,N_c=3} = 68(7)$ MeV from NNLO U(3) fits to F_π with $N_f = 4$ at $a = 0.075$ fm. (Scale set from t_0 in the chiral limit!)

Need not be the same LECs as in SU(3) ChPT beyond NLO (except for B_0).
FLAG: $F_0 = 66 - 84$ MeV, mostly from m_s =const. simulations.
$10^3 L_4 = -0.08 - 0.14$, $10^3 L_5 = 0.8 - 1.45$.

Singlet axial ward identity

$$\partial_\mu \widehat{A}^{0\mu} = \left(\overline{\psi}\gamma_5 \,\widehat{\{M, t^0\}}\, \psi\right) + \sqrt{2N_f}\widehat{q}_{\mathrm{t}}, \qquad\qquad M = \mathrm{diag}\left(m_u, m_d, m_s\right).$$

Renormalization to the $\overline{\mathrm{MS}}'$ scheme:

$$\hat{A}^{0\mu} = Z_A^s A^\mu, \qquad \hat{m}_q = Z_A^{ns}\widetilde{m}_q / Z_P^{ns} \qquad \hat{q}_{\mathrm{t}} = Z_G q_{\mathrm{t}} + Z_{GA}\partial_\mu A^{0\mu}.$$

$\widetilde{m}_q$ determined from non-singlet Ward-identities.

In general $\partial_\mu A^{0\mu}$ and q_{t} mix under renormalization [Larin,hep-ph/9302240].

q_{t} suffers from large a effects $\rightarrow q_{\mathrm{t}}^I(x) = B_G q_{\mathrm{t}}(x)$ such that $Z_G(1 + O(a)) \mapsto 1$.

For the case of $m_s = m_\ell$:

$$Z_A^s \partial_\mu A^{0\mu} = 2Z_A \frac{Z_P^s}{Z_P}\widetilde{m}_q P^0 + \sqrt{6}\left(q_{\mathrm{t}}^I + Z_{GA}\partial_\mu A^{0\mu}\right),$$

$P^0 = \frac{1}{\sqrt{3}}(u\gamma_5\bar{u} + d\gamma_5\bar{d} + s\gamma_5\bar{s})$.

Singlet axial Ward identity

Using $\sum_x \partial_\mu A^{0,\mu}(x) = 0$ for periodic b.c. or o.b. in the bulk.

$$B_G = -Z_A^{ns} \frac{Z_P^s}{Z_P^{ns}} \widetilde{m}_q \frac{\langle Q_{\mathrm{t}} \sum_x P^0(x) \rangle}{\langle Q_{\mathrm{t}}^2 \rangle}.$$

Assume $Z_P^s = Z_P^{ns}$ (non-perturbative determination ongoing).

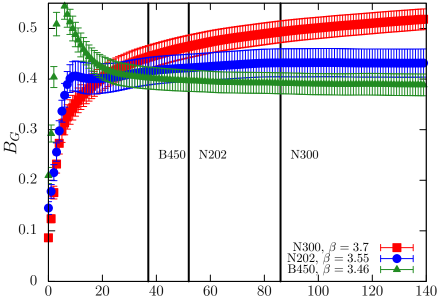
Z_{GA} then determined from Ward identity for an external state:

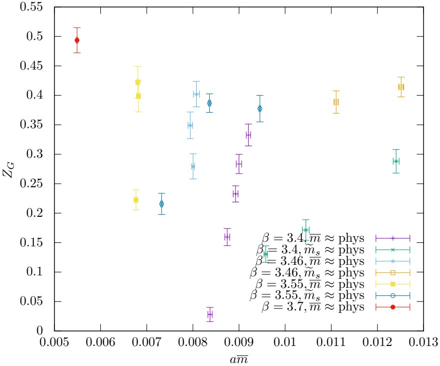
$$(Z_A^s - Z_{GA})\, \partial_\mu \langle 0|A^{0\mu}|\eta' \rangle = 2 Z_A^{ns} \frac{Z_P^s}{Z_P^{ns}} \widetilde{m}_q \langle 0|P^0|\eta' \rangle + \sqrt{6} \langle 0|q_{\mathrm{t}}^I|\eta' \rangle.$$

q_{t}, Q_{t} evaluated using gauge fields, smoothed via the Wilson flow [Lüscher,1006.4518] to a fixed flow time in physical units.

Determining B_G

$B_G = B_G(a, m_q)$. There are large m_ℓ-dependent lattice spacing effects.

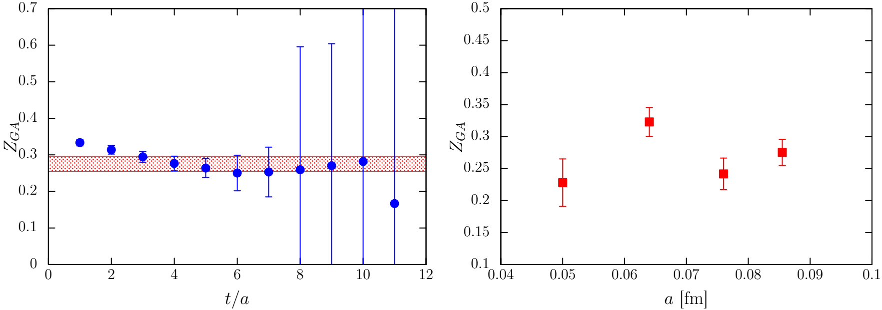




Determining Z_{GA}

Z_{GA} from $m_\ell = m_s$ ensembles.

(Determining this separately for each ensemble would have enforced the singlet AWI.)



Gluonic axial matrix element

Two possible definitions ($\mathcal{M} = \eta, \eta'$):

- direct (gluonic):

$$\langle 0 \mid \hat{q}_t | \mathcal{M} \rangle = \langle 0 \mid q_t^I | \mathcal{M} \rangle + Z_{GA} \partial_\mu \langle 0 \mid A^{0,\mu} | \mathcal{M} \rangle$$

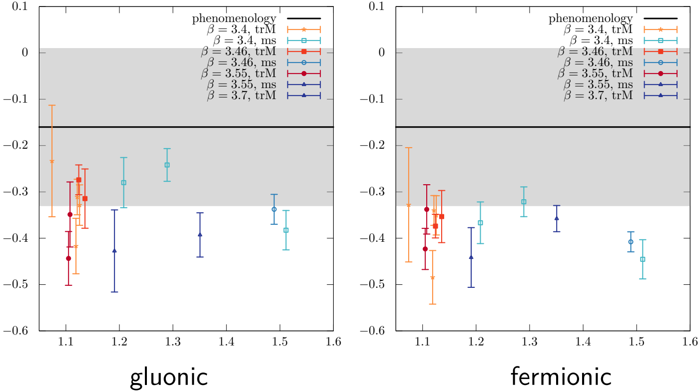
- fermionic (via AWI):

$$\sqrt{2N_f} \langle 0 \mid \hat{q}_t | \mathcal{M} \rangle = \partial_\mu \langle 0 \mid \hat{A}^{0,\mu} | \mathcal{M} \rangle - \langle 0 \mid \overline{\psi} \gamma_5 \widehat{\{M, t^0\}} \psi | \mathcal{M} \rangle$$

Z_{GA} from $m_\ell = m_s$ ensembles.

Gluonic axial matrix element

$\langle 0|\hat{q}_t|\eta'\rangle$ [GeV^3] vs. $\phi_4 \propto 2M_K^2 + M_\pi^2$



Phenomenological estimate ($\eta - \eta' - G$ mixing) [Cheng,0811.2577].

Summary:

- ⋆ Masses and decay constants of η and η' determined for a wide range of pion masses (with two trajectories to the physical point) and several lattice spacings.
  A controlled continuum and quark mass extrapolation is performed.
  Octet baryon spectrum in agreement with experiment.
- ⋆ Decay constants directly extracted from axial matrix elements.
  Similar values to those obtained using pseudoscalar matrix elements and various assumptions (probably accidental).
- ⋆ Chiral extrapolation performed using large N_c ChPT to NLO.
  LECs determined.
- ⋆ Singlet axial Ward identity works but $\exists$ large cut-off effects in the gluonic definition of the topological charge density, even after Wilson flow.
  Gluonic axial matrix elements determined.

The future:

- ⋆ Extend the analysis to more ensembles (a, m_q), in particular, along the $m_\ell = m_s$ line.
- ⋆ Look for consistency of LECs of large N_c ChPT. Fits also for F_K and F_π.