Pattern formation in sandpile models of self-organized criticality

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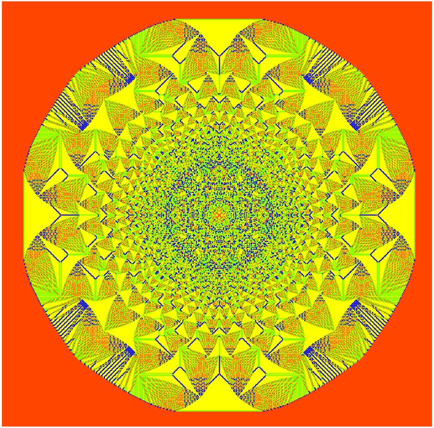
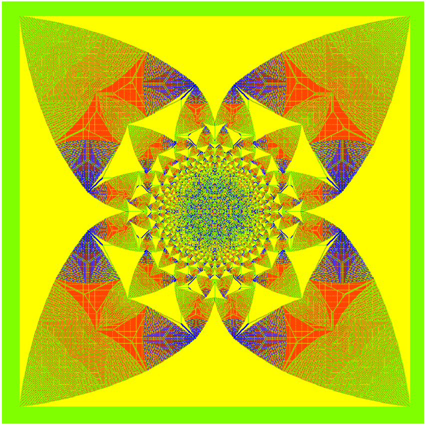
28 Dec 2010

The patterns





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- Iterate until stable .

The relaxed height distribution forms deterministic complex patterns.

The patterns

$N = 4 \times 10^4$. The color code: Red=0,Blue=1,Green=2,Yellow=3.

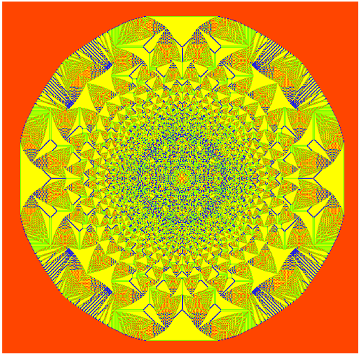


Figure: Background all $z_i = 0$

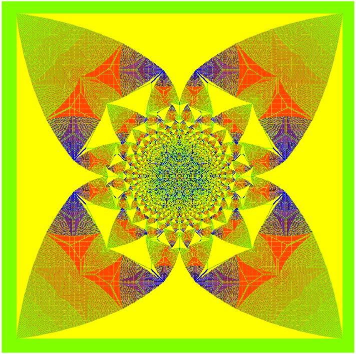


Figure: Background all $z_i = 2$

Outline

- Motivation
- Characterization of the patterns
- Robustness to external noise
- Possible connection to some interesting mathematics

Motivation

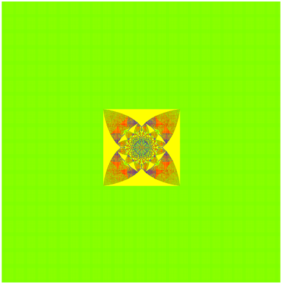


Figure: $N = 4 \times 10^4$

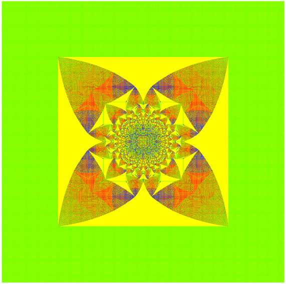


Figure: $N = 2 \times 10^5$

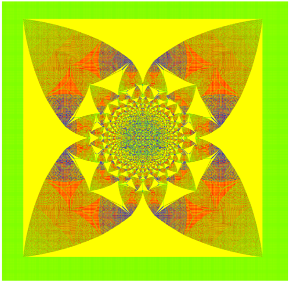
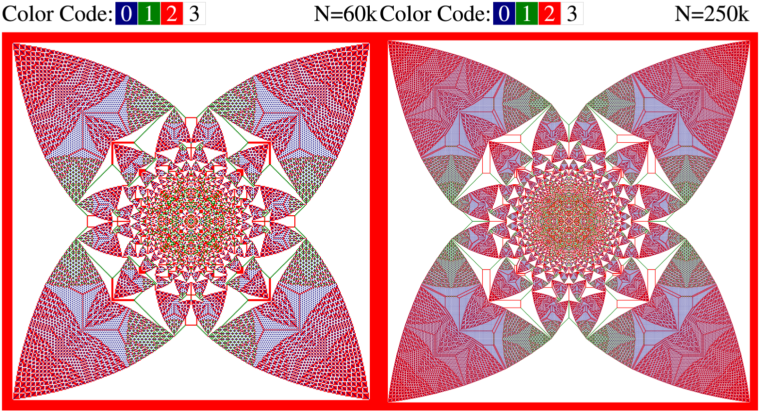


Figure: $N = 4 \times 10^5$

Diameter $\sim \sqrt{N}$.

Proportionate growth



Proportionate growth

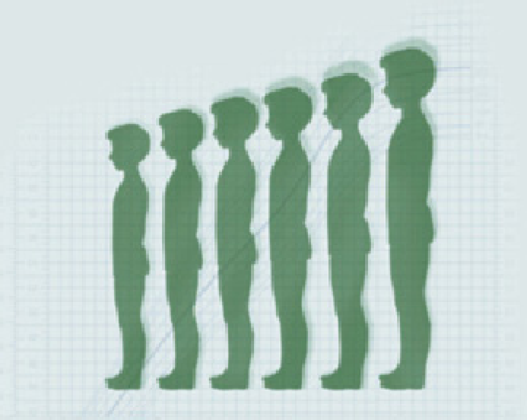
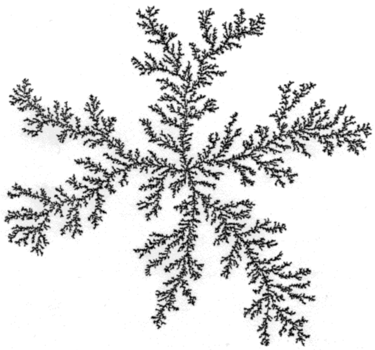


Figure: Different body parts in animals grow roughly at the same rate.

Motivation . . . contd.

- Proportionate growth requires regulation, and/or communication between different parts.
- Most existing models of growth in physics literature DLA, KPZ growth, Invasion percolation, etc does not have this property.
- Extra symmetries and robustness



Motivation . . . contd.

- Emergence of complex structures from simple local rules, e.g. Fractals
- Sandpile patterns are complex, yet simpler to analytically characterize.
- Exact characterization of the pattern involves some interesting mathematics
  - Discrete analytic functions
  - Tropical polynomials

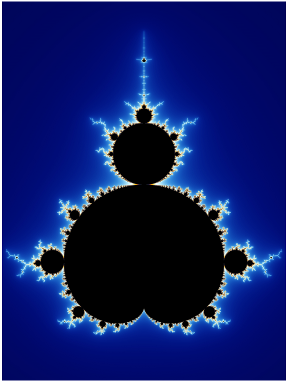


Figure: Mandelbrot set

Characterizing the pattern

- Proportionate growth + Diameter $\sim \sqrt{N}$

$\Rightarrow$ Describe in reduced coordinates

$$\xi = x/\sqrt{N}, \quad \eta = \mathrm{y}/\sqrt{\mathrm{N}}$$

- Characterize pattern in terms of density of heights $\rho(\xi, \eta)$ $=$ the height averaged over an area $\delta\xi\delta\eta$ around (ξ, η), with $1/\sqrt{N} \ll \delta\xi \ll 1$ and $1/\sqrt{N} \ll \delta\eta \ll 1$.

Characterizing the pattern

Let $T(x,y) = \#$ of toppling at (x,y).

$$\sum' T(x',y') - 4T(x,y) = \Delta z(x,y) - N\delta_{x,0}\delta_{y,0} \qquad (1)$$

Define

$$\phi(\xi,\eta) = \lim_{N\to\infty} \frac{T(x,y)}{N}$$

Then

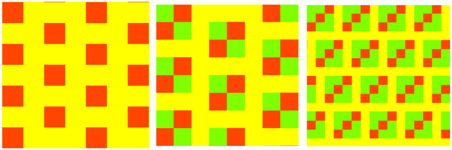
$$\boxed{\nabla^2\phi(\xi,\eta) = \Delta\rho(\xi,\eta) - \delta(\xi)\,\delta(\eta)\,,}$$

where $\Delta\rho$ is the change in density.

The complete specification of ϕ determines the patterns.

Determining ϕ

- Patches with periodic heights.



▸ pattern

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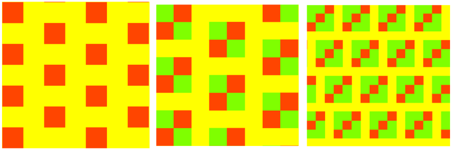

pattern

- $\rho(\xi, \eta)$ is constant within a patch.

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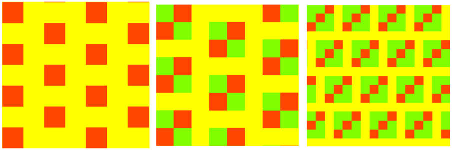

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- $\rho(\xi, \eta)$ is constant within a patch.
- Lemma:

ϕ is a quadratic function of ξ, η in each patch. ▸ Proof

Determining ϕ

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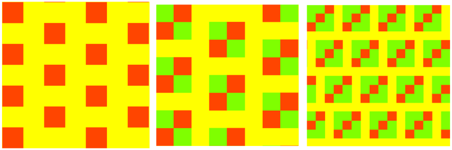

pattern

- $\rho(\xi, \eta)$ is constant within a patch.
- Lemma:

$$\boxed{\phi \text{ is a quadratic function of } \xi, \eta \text{ in each patch.}}$$

Proof

- Continuity of ϕ and its first derivatives along the patch boundaries imposes constraints.

Solve the constraints and determine ϕ.

Simpler pattern

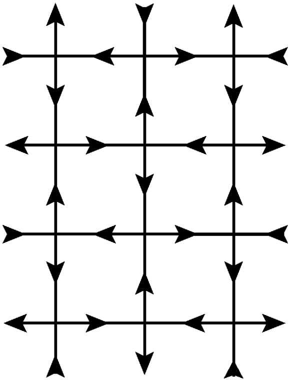

Figure: F-lattice with $z_c = 2$

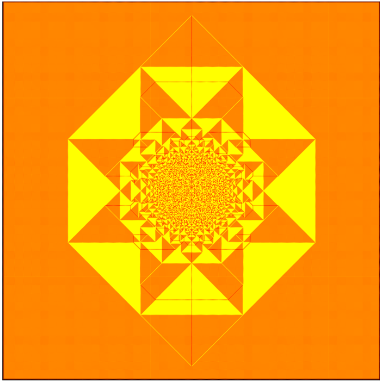

Figure: $N = 2x10^5$ on checkerboard background of 1 and 0 heights

Adjacency graph

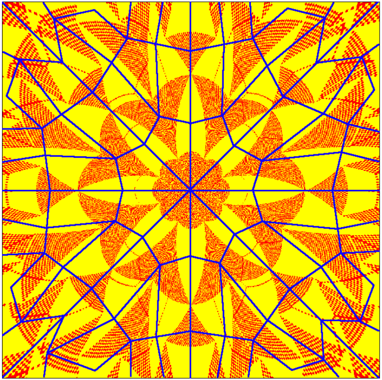

Figure: Adjacency graph

1/z2 picture

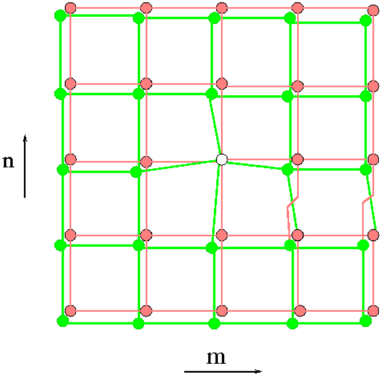

Figure: Representation as a square lattice on two sheeted Riemann surface

Quantitative characterization

- Label patches using (m, n).

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$$\phi(\xi,\eta) = \frac{1}{8}(m+1)\xi^2 + \frac{1}{4}n\xi\eta + \frac{1}{8}(1-m)\eta^2 + d\xi + e\eta + f$$

In a light patch

$$\phi(\xi,\eta) = \frac{1}{8}m\xi^2 + \frac{1}{4}n\xi\eta - \frac{1}{8}m\eta^2 + d_{m,n}\xi + e_{m,n}\eta + f_{m,n}$$

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In a light patch

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- Continuity of ϕ and its first derivatives along the patch boundaries imply, $d_{m,n}$ and $e_{m,n}$ follows

$$\psi_{m+1,n+1} + \psi_{m+1,n-1} + \psi_{m-1,n+1} + \psi_{m-1,n-1} - 4\psi_{m,n} = 0,$$

Quantitative characterization

Boundary condition:

- $d(0,0) + ie(0,0) = 0$
- For large $|m + in|$,

$$d(m,n) + ie(m,n) \simeq \pm \frac{1}{\sqrt{2\pi}} \sqrt{m + in}$$

Solve this set of linear equations numerically on a large grid.

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Solve this set of linear equations numerically on a large grid.

The pattern has exact eight fold rotational symmetry. ▸ Aside

Multiple sites of addition

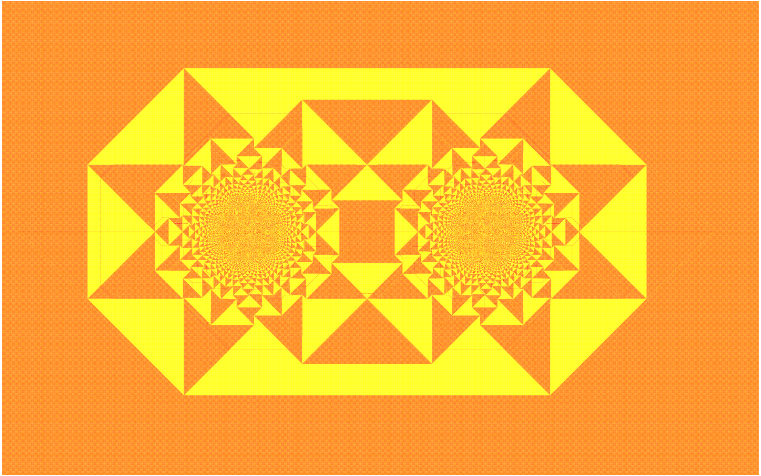


Figure: $N = 10^5$ added each at sites $(-400, 0)$ and $(400, 0)$

Adjacency graph

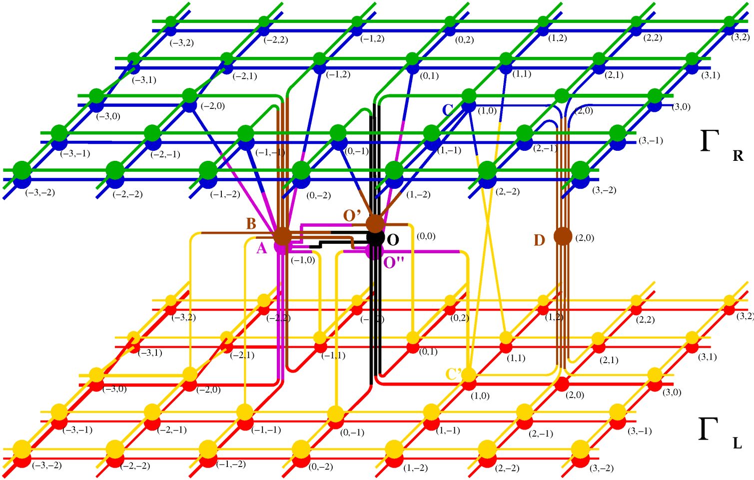


Figure: Adjacency graph

Line of sink sites

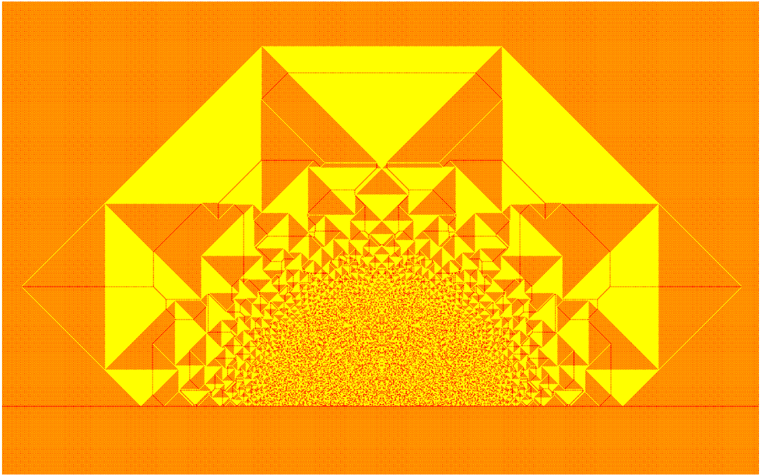


Figure: $N = 10^5$ added at site $(0, 1)$ with sink sites along the x-axis.

Adjacency graph

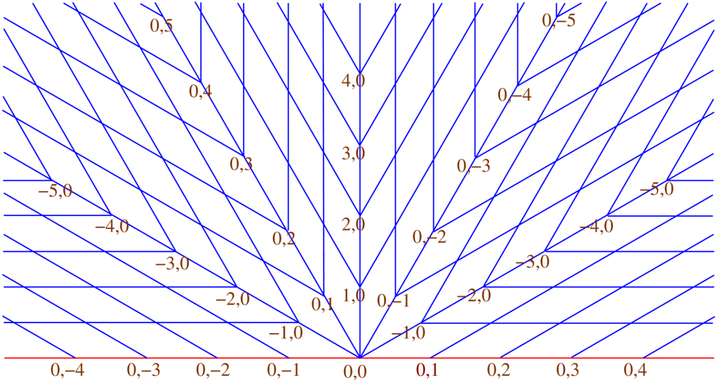


Figure: Adjacency graph

Growth rate

"Diameter" of the pattern

$$\Lambda \sim N^{\alpha}$$

In absence of sink sites $\alpha = 1/2$.
For the line sink:

- # un-absorbed particles $N_r \sim \Lambda^2$
- # absorbed particles $N_a \sim \Lambda^2 \int_{1/\Lambda}^{1} d\xi \frac{\partial}{\partial \eta} \phi \Big|_{\eta=0}$
- Close to the sink line

$$\phi \sim \frac{\cos(\theta)}{r} \Rightarrow N_a \sim \Lambda^3$$

- Hence,

$$\boxed{C_1\Lambda^3 + C_2\Lambda^2 = N} \Rightarrow \Lambda \sim N^{1/3} \text{ For large N.}$$

Growth rate

- The equation gives correction to scaling.
- Unexpected accuracy: For $C_1 = 0.1853$, and $C_2 = 0.528$ the solution of this equation differs from the actual $\Lambda(N)$ by at most 1, for $100 < N < 3 \times 10^6$

Other sink geometries

- For a wedge of θ, $\Lambda \sim N^{\alpha}$, with $\alpha = 1/(2 + \pi/\theta)$.
- For a point sink adjacent to the site of addition $\Lambda \sim \sqrt{N/\log N}$.
- Generalizable to higher dimensions

Fast growing sandpiles

- If the initial background density is low enough everywhere,

$$\boxed{\Lambda \sim N^{1/2}}$$

- If many sites have large height

$$\boxed{\Lambda = \infty \quad \text{for finite N}}$$

- For an in-between set of periodic backgrounds

$$\boxed{\Lambda \sim N^{\alpha} \quad \text{for } 1/2 < \alpha \leq 1}$$

Lemma: The potential function

$$\phi(\xi, \eta) = Lim_{N\to\infty} \frac{1}{N} T(N^{\alpha}\xi, N^{\alpha}\eta)$$

for fast-growing sandpiles ϕ is linear inside periodic patches.

Proof: Proof as before.

Pattern on F-lattice showing $\alpha = 0.55$

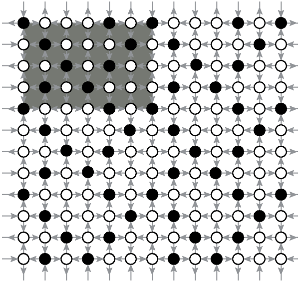


Figure: Periodic background: Filled circle=0, unfilled=1

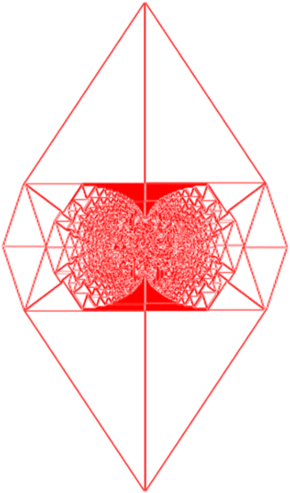


Figure: Only patch boundaries are drawn

Directed Triangular lattice showing $\alpha = 1$

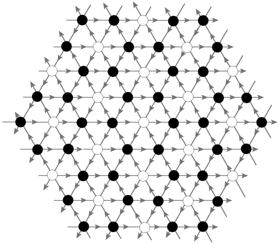


Figure: Periodic background: Filled circle=1, unfilled=2

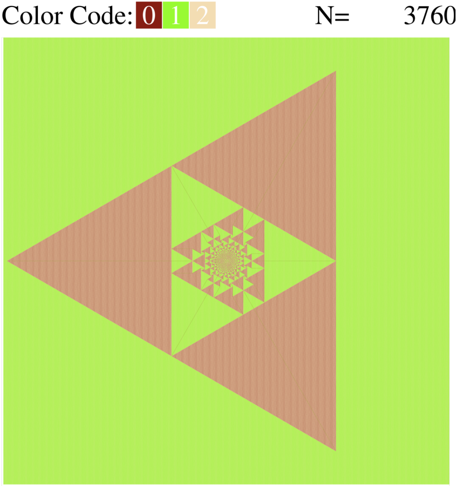
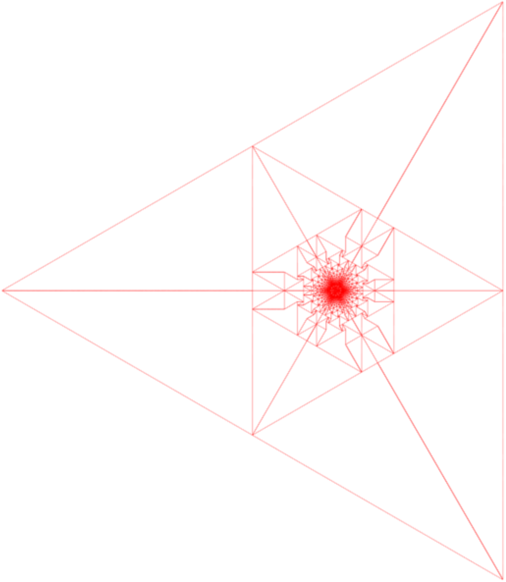
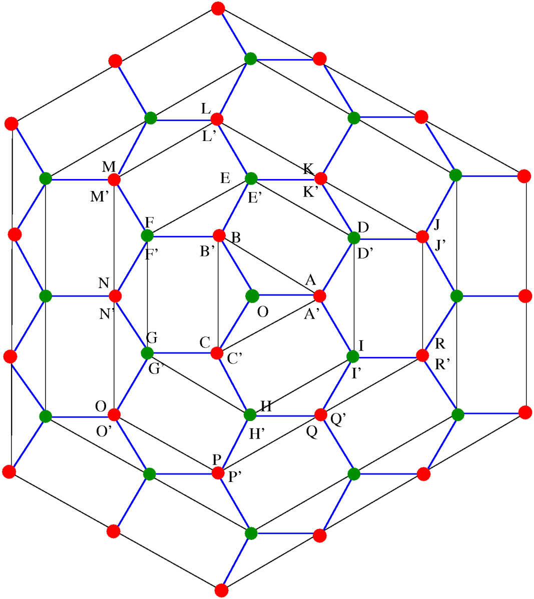


Figure: Patern with $N = 1000$



The adjacency graphs



Characterizing the triangular non-compact pattern

Analysis is similar to the earlier characterization, actually simpler.

The potential function in different patches is given by

$$\phi_P = a_p \, \xi + b_P \, \eta + f_p$$

a_P and b_p are determined by matching slope discontinuity to line charge densities.

Then, f_p satisfies a Laplace's equation on the adjacency graph.

Robustness of the pattern

The arguments only depend on the existence of only two types of patches, and straight line boundaries.

These can be found (by trial and error) in other cases also.
Then the asymptotic pattern is identical.
Some examples:

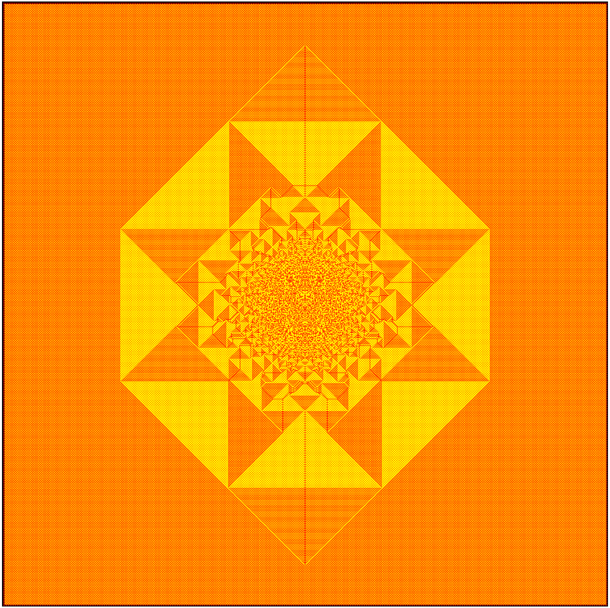


Figure: F-lattice with background density 5/8

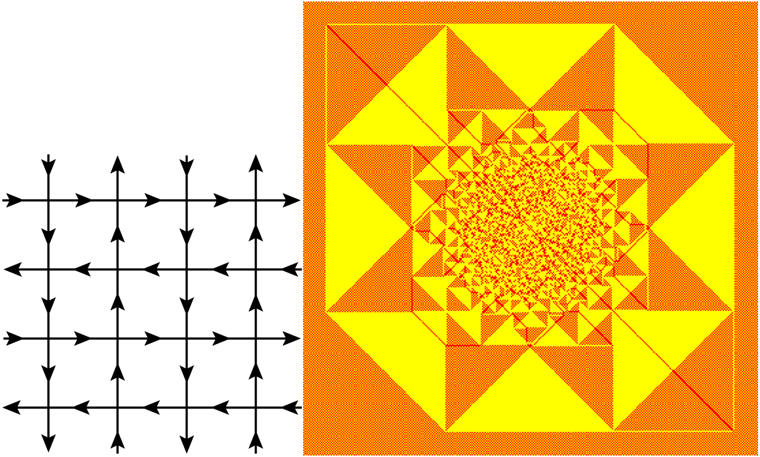


Figure: Manhattan lattice, with initial density $1/2$, and $120,000$ particles

Robustness

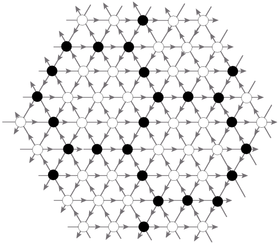


Figure: Periodic background: Filled circle=1, unfilled=2

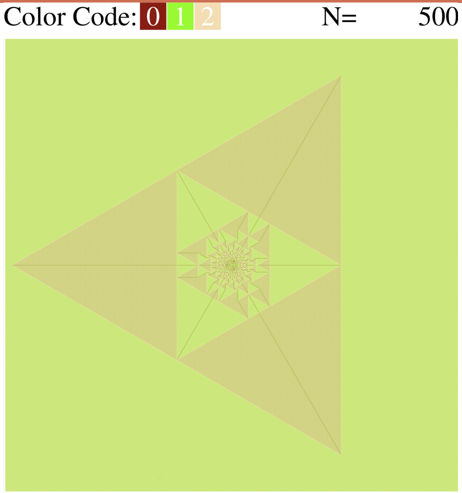


Figure: Pattern with $N = 1000$

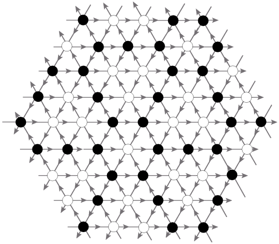


Figure: Periodic background: Filled circle=1, unfilled=2

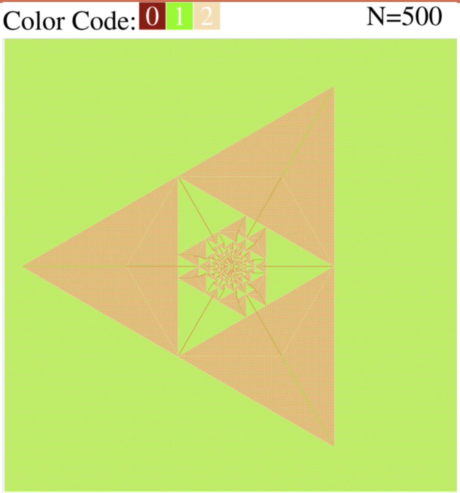


Figure: Pattern with $N = 1000$

Robustness

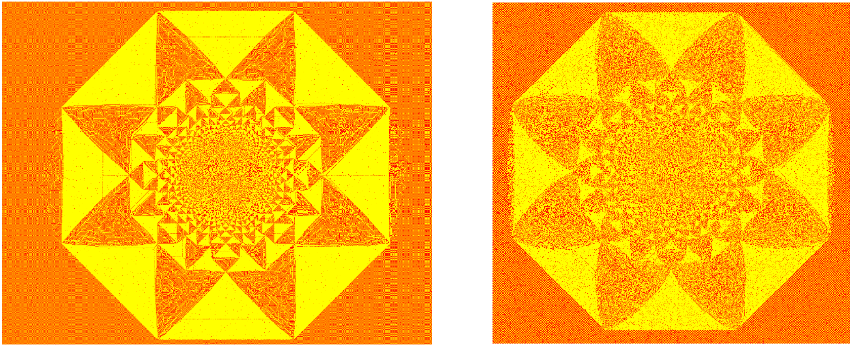
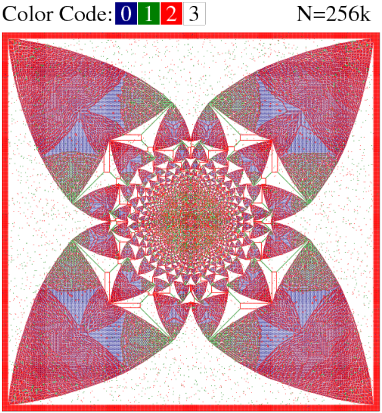


Figure: (a) 1% noise (b) 10%

Noise in the initial particle distribution.

Noise in the background



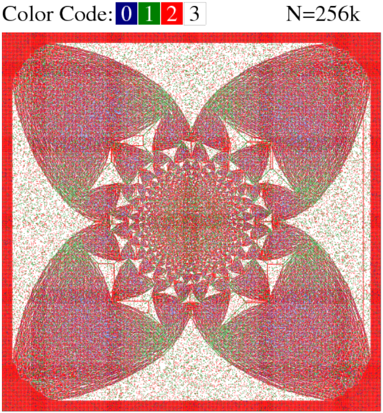


Figure: (a) 1% noise (b) 10%

Robustness

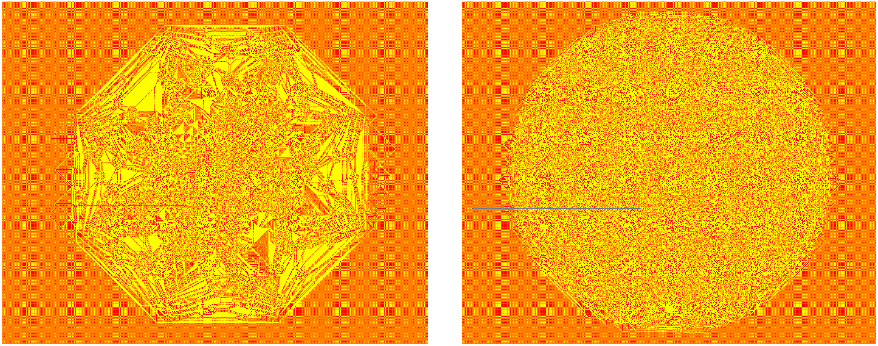


Figure: (a) 0.1% noise (b) 1%

Noise in the relaxation rule.

Random broken edges

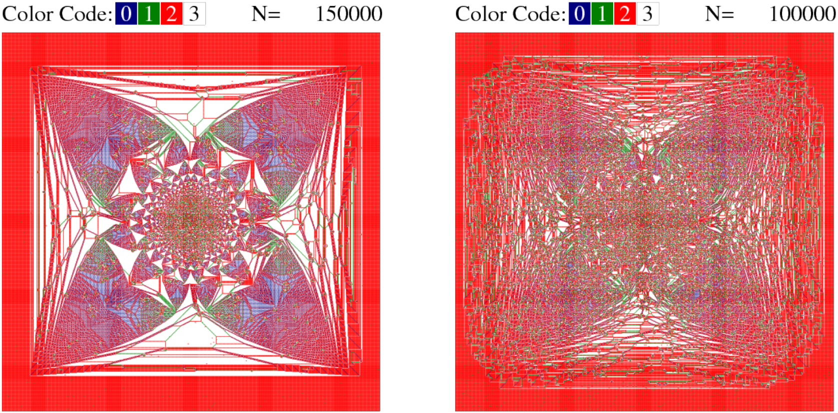


Figure: (a) 1% noise (b) 10%

A variational formulation

The general principle is called the (lazy man's)

'Least Action Principle': The actual pattern is the stable pattern reached by minimum number of toppling.

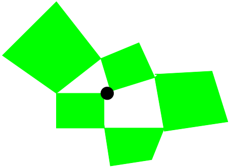
$$\nabla^2\phi = +\delta\rho - \delta(\xi,\eta)$$

Proof is trivial for abelian models: If a site is unstable, it will not stablize, until toppled. Order of toppling does not matter.

Formulation as an electrostatics problem

We have $\nabla^2\phi = +\delta\rho - \delta(\xi,\eta)$
Positive point charge $+1$ at origin, and unit negative charge of areal density 1

Can we distribute the negative charge in such a way that the net potential is piecewise-quadratic, and exactly zero far away?



The answer, presumably unique, is the observed pattern on the F-lattice.
Other backgrounds have more choices of charge densities .

Discrete Quadratic Approximants

Example of discrete approximants:

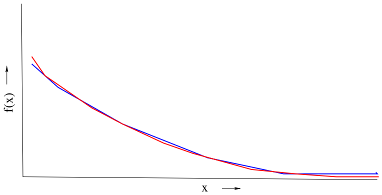


Figure: Approximate $f(x)$ by piece-wise linear functions with integer slopes

The best "discrete approximant" to a given smooth function.

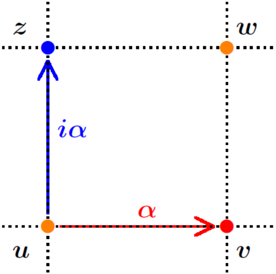
An iterative formulation

- Start with a trial pattern.
- Determine the corresponding $\phi(\xi, \eta)$
- Determine the "best" piece-wise quadratic approximants to $\phi(\xi, \eta)$ using the given set of quadratic functions ϕ_P.
- The correspond charge density is piece-wise constant. Remove singularities at boundaries.
- Determine corresponding potential $\phi^{(1)}(\xi, \eta)$.
- Iterate

If the process converges, we get the asymptotic pattern.

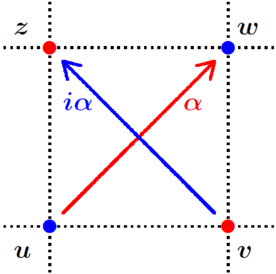
Discrete Analytic Functions

Functions defined only on discrete points in the complex $z-$ plane.



$$\frac{F(z)-F(u)}{z-u} = \frac{F(v)-F(u)}{v-u} \quad (1^{st})$$

$$F(z) - F(u) = i\,(F(v) - F(u))$$



$$\frac{F(z)-F(v)}{z-v} = \frac{F(w)-F(u)}{w-u} \quad (2^{nd})$$

$$F(z) - F(v) = i\,(F(w) - F(u))$$

simple discrete analytic functions are constant, z, z^2, z^3, $z^4 - z\bar{z}, \ldots$

Define DA function $F_{1/2}(z)$, which varies as $\sqrt{z}$ for large $|z|$, and $F(0) = 0$

The function $d(m,n) + ie(m,n)$ which characterizes the pattern for F-lattices is $cF_{1/2}(m + in)$.

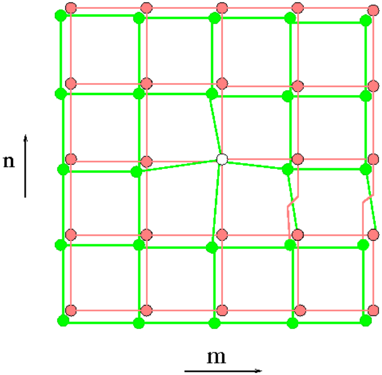


Figure: A discretized two sheeted Riemann surface for $F_{1/2}(z)$

Connection to Tropical Mathematics

Define

$$a \oplus b = Max[a, b]$$

$$a \otimes b = a + b$$

Then standard properties of usual addition and multiplication (commutative, identity, distributive ..) contiue to hold.

Example: $3 \oplus 5 \oplus 2 = 5$

$3 \otimes 4 = 7$

Tropical polynomials: $a \otimes x \otimes x \oplus b \otimes x \oplus c$

Example: $x \otimes x \oplus 2 \otimes x \oplus 5 = Max[2x, x + 2, 5]$.

Fundamental theorem of tropical algebra.

A piecewise -linear convex function can be represented as a tropical polynomial.

Hence useful for describing the function $\phi(\xi, \eta)$.

Summary

- A model of proportionate growth
- Quantitatively characterized a large class of patterns with only two types of patches.
- Additional symmetries.
- Characterized patterns with multiple sources and sinks. Also determined the growth rates
- Analyzed a large class of patterns with $\Lambda > \sqrt{N}$ and quantitatively characterized some such patterns.

References

- Pattern formation in growing sandpiles.
  Deepak Dhar, Tridib Sadhu and Samarth chandra, *Euro. Phys. Lett.*,(2008).
- Pattern formation in growing sandpiles with multiple sources and sinks.
  Tridib Sadhu and Deepak Dhar,*J. Stat. Phys.*,(2009).
- Pattern formation in fast growing sandpiles.
  In preparation.

Thank you

Conformal transformation

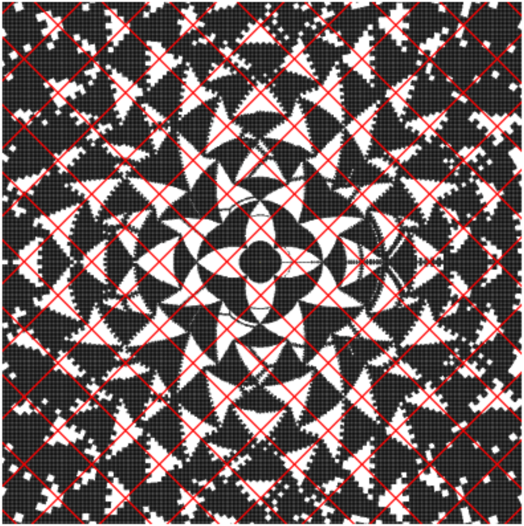


Figure: $z' = 1/z^2$ picture of the original picture. goback

Proof

Proof: Taylor expand $\phi(\xi, \eta)$ inside a patch about (ξ_0, η_0).

$$\phi(\xi_0+\Delta\xi, \eta_0+\Delta\eta) = \phi(\xi_0, \eta_0)+d\Delta\xi+e\Delta\eta+a_2\Delta\xi^2+\cdots+K(\Delta\xi)^3+\cdots$$

In terms of toppling number function $T(X, Y)$ this becomes

$$T(X_0 + \Delta X, Y_0 + \delta Y) \simeq T(X_0, Y_0) + d\sqrt{N}\Delta X + e\sqrt{N}\Delta Y + a_2\Delta X^2 + \cdots + \frac{K(\Delta X)^3}{\sqrt{N}} + \cdots$$

Since T is always an integer, it would jump by 1 at separations $N^{1/6}$, causing many defect lines. Hence $K = 0$. main stream

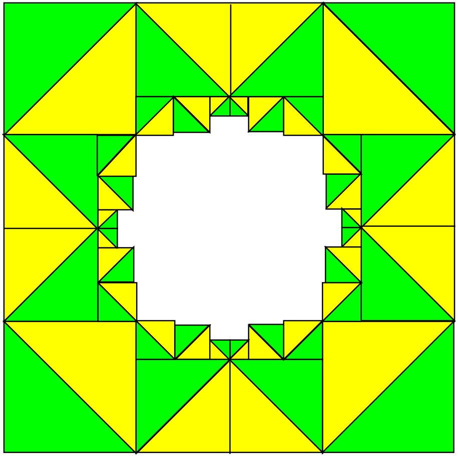


Figure: Tiling with square tiles

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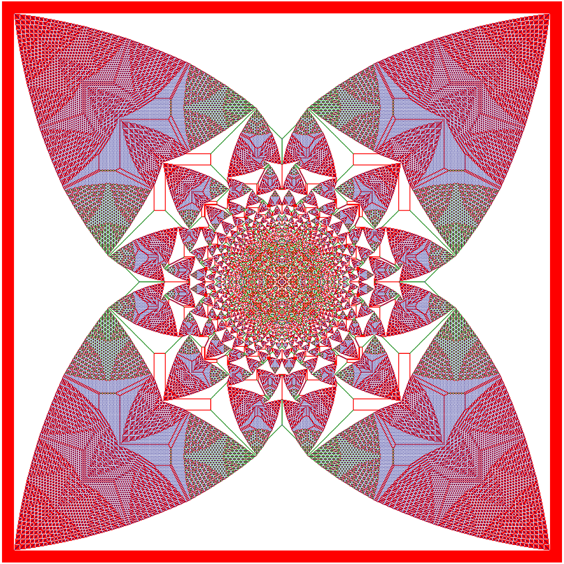
Color Code: 0 1 2 3 N=250k



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