Error propagation and systematic uncertainties

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Overview

Error propagation master formulae

- Consider a function of n random variable $y = y((x_1, x_2, ..x_n))$ and we know $\vec{X} = (x_1, x_2, ..x_n)$ distributed according to a joint p.d.f $f(\vec{X})$, then

$$\bar{y} = E[y(\vec{X})] = \int y(\vec{X}) f(\vec{X}) d\vec{X}$$

,

$$\sigma_y^2 = E[(y(\vec{X}) - \bar{y})^2]$$

- If we don't have proper knowledge of $f(\vec{X})$ can we measure σ_y?
- Yes, under certain assumption,
  - We know mean values of x_i which is $\bar{\vec{X}} = \vec{\mu}$
  - $y(\vec{X})$ is linear nearby $\vec{\mu}$ ($\left[\frac{\partial^k y}{\partial x_i^k}\right]_{\vec{x}=\vec{\mu}} << \left[\frac{\partial y}{\partial x_i}\right]_{\vec{x}=\vec{\mu}}$ for $k > 1$, any other cross term too)
  - We have knowledge on correlation matrix $V_{ij} = E[x_i x_j - E(x_i x_j)]$
- Under above assumptions we can derive the master formulae,

$$\sigma_y^2 = \sum_{i,j}^{n} \left[\frac{\partial y}{\partial x_i} \frac{\partial y}{\partial x_j}\right]_{\vec{X}=\vec{\mu}} V_{ij} \tag{1}$$

How to derive? Not so important

$$y(\vec{x}) \approx y(\vec{\mu}) + \sum_{i=1}^{n} \left[\frac{\partial y}{\partial x_i}\right]_{x=\vec{\mu}} (x_i - \mu_i) \tag{2}$$

$$E[y(\vec{x})] \approx y(\vec{\mu}) \tag{3}$$

$$E[y^2(\vec{x})] \approx y^2(\vec{\mu}) + 2y(\vec{\mu}) \sum_{i=1}^{n} \left[\frac{\partial y}{\partial x_i}\right]_{x=\vec{\mu}} E(x_i - \mu_i) + E\left[\left(\sum_{i=1}^{n} \left[\frac{\partial y}{\partial x_i}\right]_{x=\vec{\mu}} (x_i - \mu_i)\right)\left(\sum_{j=1}^{n} \left[\frac{\partial y}{\partial x_j}\right]_{x=\vec{\mu}} (x_j - \mu_j)\right)\right]$$

$$\approx y^2(\vec{\mu}) + \sum_{i,j=1}^{n} \left[\frac{\partial y}{\partial x_i}\frac{\partial y}{\partial x_j}\right]_{\vec{X}=\vec{\mu}} V_{ij}$$

Some special cases

- $y = x_1 + x_2 \implies \sigma_y^2 = \sigma_1^2 + \sigma_2^2 + 2V_{12}$
  - The 'absolute' uncertainty of addition of n uncorrelated $(V_{ij} = 0)$ variables will be square root of quadrature sum of 'absolute' uncertainty of individual variables
- $y = \frac{x_1 x_2}{x_3 x_4} \implies \frac{\sigma_y^2}{y^2} = \frac{\sigma_1^2}{x_1^2} + \frac{\sigma_2^2}{x_2^2} + \frac{\sigma_3^2}{x_3^2} + \frac{\sigma_4^2}{x_4^2} + (2\frac{V_{12}}{x_1 x_2} +$ and other 5 correlation terms)
  - If the uncorrelated variables of a observable, are associated to a formulae with only multiplication or division form, then 'relative error' of the observable add up in quadrature
- If two or more variable's correlation coefficient $\rho_{ij} = \frac{V_{ij}}{\sigma_i \sigma_j}$ is one, which means they are 100% correlated then
  - $y = x_1 + x_2 \implies \sigma_y = \sigma_1 + \sigma_2$
  - $y = \frac{x_1 x_2}{x_3 x_4} \implies \frac{\sigma_y}{y} = \frac{\sigma_1}{x_1} + \frac{\sigma_2}{x_2} + \frac{\sigma_3}{x_3} + \frac{\sigma_4}{x_4}$
  - As an example suppose we assign $x\%$ relative systematic due to data/MC efficiency correction for one charged track. Then if there are m charged tracks in signal then we can assign $mx\%$ relative systematics. This works because of underlying assumption Track finding efficiency for the first track is 100% correlated with rest of track finding efficiency.

Let's try for others

- $g = \frac{4\pi^2 \ell}{T^2}$
- $L(T) = aT^2 + bT + c$

and so on..

Systematic uncertainty of MC signal efficiency

- Signal efficiency is a essential input to measure several physical observable. This is generally estimated from signal MC. Now due to imperfect simulation, then efficiency for data and MC are not exactly same. This introduce systematic uncertainty called as multipicative systematic uncertainty.
- Let's say,

$$\epsilon_{\mathrm{MC}} = \epsilon_0 \times \epsilon_{\mathrm{pid}}^{\mathrm{mc}} \times \epsilon_{\mathrm{track}}^{\mathrm{mc}} \times ...\epsilon_n^{\mathrm{mc}},$$

  where ϵ_0 is the efficiency when no selection is applied. $\epsilon_{\mathrm{pid}}^{\mathrm{mc}}$ is efficiency when pid selection applied and so on.. Similarly, for data we have for data,

$$\epsilon_{\mathrm{data}} = \epsilon_0 \times \epsilon_{\mathrm{pid}}^{\mathrm{data}} \times \epsilon_{\mathrm{track}}^{\mathrm{data}} \times ...\epsilon_n^{\mathrm{data}},$$

  So,

$$\epsilon_{\mathrm{data}} = \epsilon_{\mathrm{MC}} \times r_{\mathrm{pid}} \times r_{\mathrm{track}} \times ...r_{\mathrm{n}},$$

  where $r_n = \epsilon_n^{\mathrm{data}}/\epsilon_n^{\mathrm{mc}}$
- So assuming sources of systematics causing data mc difference are uncorrelated,

$$\frac{\sigma_{\epsilon_{\mathrm{data}}}}{\epsilon_{\mathrm{data}}} = \sqrt{\left(\frac{\sigma_{\epsilon_{\mathrm{MC}}}}{\epsilon_{\mathrm{MC}}}\right)^2 + \left(\frac{\sigma_{r_{\mathrm{pid}}}}{r_{\mathrm{pid}}}\right)^2 + \left(\frac{\sigma_{r_{\mathrm{track}}}}{r_{\mathrm{track}}}\right)^2 + ... \left(\frac{\sigma_{r_{\mathrm{n}}}}{r_{\mathrm{n}}}\right)^2}$$

Continued...

- $\sigma_{\epsilon_{\mathrm{MC}}}/\epsilon_{\mathrm{MC}}$
  - Due to limited MC: $\sqrt{\epsilon_{\mathrm{MC}}(1-\epsilon_{\mathrm{MC}})/N}$ where N is number of generated sample in MC. This is inspired by the relative error on a binomially distributed random variable.
  - Some times we assume some pdg BF to measure ϵ_{MC}. As an example we need $\epsilon_{B^0\to J/\psi[\to e^+e^-]K_s^0}$, but we only reconstruct $K_s^0 \to \pi^+\pi^-$ in the analysis. In this scenario, $\epsilon_{B^0\to J/\psi[\to e^+e^-]K_s^0} = \frac{\epsilon_{B^0\to J/\psi[\to e^+e^-]K_s^0[\pi^+\pi^-]}}{\mathrm{BF}(K_s^0\to\pi^+\pi^-)}$.
    We can generate exclusive decay sample $B^0 \to J/\psi[\to e^+e^-]K_s^0[\pi^+\pi^-]$ and find efficiency in above way, or we can fix $\mathrm{BF}(K_s^0 \to \pi^+\pi^-)$ in decay file to generate. In either case we have to add this systematic uncertainty as $\sigma_{\mathrm{BF}}/\mathrm{BF}$.
- Suppose we have three charged track in the signal, then $r_{\mathrm{track}} = r_{\mathrm{track}_1} \times r_{\mathrm{track}_2} \times r_{\mathrm{track}_3}$ and $\sigma_{r_{\mathrm{track}}}/r_{\mathrm{track}} = 3 \times \sigma_r/r^3$ where r and σ_r are average data/MC track finding efficiency and its uncertainty respectively calculated from standard control sample. These value are generally provided from performance group.

Systematic uncertainty associated with fit Yield (N_s)

- PDF shape: Sometimes we fix some fit parameters from MC, and we introduce systematic uncertainty for that. $(\Delta\text{Yield}/\text{Yield})^2 = \sum_{i=1}^{k} \left(\frac{\partial \text{Yield}}{\partial s_i}\right)^2 \sigma_{s_i}^2$ assuming the parameters fixed from MC are **uncorrelated**. This formula can be modified to $(\Delta\text{Yield}/\text{Yield})^2 = \sum_{i=1}^{k} \left(\frac{\Delta\text{Yield}_i}{\Delta s_i/\sigma_{s_i}}\right)^2$. Common practice is to vary s_i by $s_i \rightarrow s_i \pm \sigma_{s_i}$ such that $\Delta s_i/\sigma_{s_i} = 1$ and find the change Yield_i , which leads to $(\Delta\text{Yield}/\text{Yield})^2 = \sum_{i=1}^{k} (\Delta\text{Yield}_i)^2$

There are be other source of systematic like cut variation, selection, parametrization, impact of specific backgrounds, detector resolution effect, which in general can be added in qudrature.

Systematic uncertainty of BF measurement

$$\text{BF} = \frac{N_s}{2 \times \epsilon \times f^{\pm/00}(f^{00/\pm}) \times N_{B\bar{B}}}$$

Since all of them are in multiplication or division format their relative error will be added in qudrature,

$$\frac{\sigma_{\text{BF}}}{BF} = \sqrt{(\sigma_{N_s}/N_s)^2 + (\sigma_\epsilon/\epsilon)^2 + (\sigma_f/f)^2 + (\sigma_{N_{B\bar{B}}}/N_{B\bar{B}})^2}$$

- Relative error on $N_{B\bar{B}}$ are generally available from standard analysis
- PDG value can be used for relative error on $f_{\pm/00}$
- Relative error on efficiency (ϵ) are discussed earlier
- Relative error on N_s (fit Yield) are discussed earlier

Systematic uncertainty on R_K measurements

$$R_K = \frac{\mathrm{BF}(B^+ \to K^+\mu^+\mu^-)}{\mathrm{BF}(B^+ \to K^+e^+e^-)} = \frac{\frac{N_{K^+\mu\mu}}{\epsilon_{K^+\mu\mu}}}{\frac{N_{K^+ee}}{\epsilon_{K^+ee}}}$$

Now,

$$\epsilon^{\text{data}}_{K\mu\mu}/\epsilon^{\text{data}}_{Kee} = \epsilon^{\text{mc}}_{K\mu\mu}/\epsilon^{\text{mc}}_{Kee} \times (r^{\text{tracking}}_{K\mu\mu}/r^{\text{tracking}}_{Kee}) \times (r^{\text{kID}}_{K\mu\mu}/r^{\text{kID}}_{Kee}) \times (r^{\text{muonID}}_{K\mu\mu}/r^{\text{electroID}}_{Kee}) \times (r^{\text{MVA}}_{K\mu\mu}/r^{\text{MVA}}_{Kee})....$$

- r ratios for tracking is unambiguously same for both mode, so this term will not be there
- r ratios for kID is also same for both mode, since this data/MC correction for both mode is same.
- If we would calculate R_{K^0} but we reconstruct $K^0 \to K_s \to \pi^+\pi^-$, and hence to calculate MC efficiency ($\epsilon_{B^0\to K^0\ell\ell}$) we use BF for neutral kaon decay to two pion. But this is same for both lepton mode, and hence this factor will not affect R_{K^0} measurement.
- Similarly factors affect data/MC efficiency for both the modes in unambiguously same way, will not contribute systematic uncertainty to R_K

$$(\sigma_{R_K}/R_K)^2 = (\sigma_{N_{K\mu\mu}}/N_{K\mu\mu})^2 + (\sigma_{N_{Kee}}/N_{Kee})^2 + (\sigma_{r_{\text{electronID}}}/r_{\text{electronID}})^2 + (\sigma_{r_{\text{muonID}}}/r_{\text{muonID}})^2$$

$$+(\sigma_{r_{\text{MVA}}}/r_{\text{MVA}})^2 + (\sigma_{\epsilon^{\text{mc}}_{K\mu\mu}}/\epsilon^{\text{mc}}_{K\mu\mu})^2 + (\sigma_{\epsilon^{\text{mc}}_{Kee}}/\epsilon^{\text{mc}}_{Kee})^2$$

Systematic uncertainty on A_I

$$A_I = \frac{2 \times (\tau_{B^+}/\tau_{B^0})(f_{\pm}/f_{00})(N_{\text{sig}}/\epsilon)\mid_{K_s^0\ell\ell} - (N_{\text{sig}}/\epsilon)\mid_{K^+\ell\ell}}{2 \times (\tau_{B^+}/\tau_{B^0})(f_{\pm}/f_{00})(N_{\text{sig}}/\epsilon)\mid_{K_s^0\ell\ell} + (N_{\text{sig}}/\epsilon)\mid_{K^+\ell\ell}}$$

The equation can be re-arranged as

$$A_I = \frac{2\tau f N_1/\epsilon_1 - N_2/\epsilon_2}{2\tau f N_1/\epsilon_1 + N_2/\epsilon_2}$$

If we use error-propagation master formula we can derive,

$$\sigma_{A_I} = \frac{4\tau f N_1/\epsilon_1 N_2/\epsilon_2}{(2\tau f N_1/\epsilon_1 + N_2/\epsilon_2)^2} \sqrt{\sum_i (\sigma_{s_i}/s_i)^2}$$

where s_i stands for N_1,N_2,ϵ_1,ϵ_2, τ, f for different i

- Relative error on fit yield N_1,N_2 are discussed earlier
- Relative error on τ, f rely on PDG value
- If we estimate A_I for lepton flavours individually then, relative error on signal efficiency is previously discussed
- If we want to estimate A_I for lepton flavour-independent way we have two different way,
  - Take weighted average of lepton flavour dependent result
  - Calculate combined yield from fit and combined efficiency from MC. Systematic uncertainty for combined efficiency is bit complicated (backup).

$$\sigma_{A_I} = \frac{4\tau f N_1/\epsilon_1 N_2/\epsilon_2}{(2\tau f N_1/\epsilon_1 + N_2/\epsilon_2)^2} \sqrt{\sum_i (\sigma_{s_i}/s_i)^2}$$

This expression contains ϵ for both modes.
$\epsilon_{\text{data}}^{K_s^0 ee} = \epsilon_{\text{mc}}^{K_s^0 ee} \times f_{\text{tracking}} \times f_{\text{electronID}} \times f_{K_s^0}$
$\epsilon_{\text{data}}^{K^+ ee} = \epsilon_{\text{mc}}^{K^+ ee} \times f'_{\text{tracking}} \times f_{\text{electronID}} \times f_{\text{kaonID}}$

- $f_{\text{electronID}}$ will cancel from A_I expression.
- $f_{\text{tracking}} = f^{\pi^+}_{\text{tracking}} f^{\pi^-}_{\text{tracking}} f^{e^+}_{\text{tracking}} f^{e^-}_{\text{tracking}}$ and $f'_{\text{tracking}} = f^{K^+}_{\text{tracking}} f^{e^+}_{\text{tracking}} f^{e^-}_{\text{tracking}}$. Hence $f^{e^\pm}_{\text{tracking}}$ cancels in the final expression of A_I.

So finally the s_i contains N_1, N_2, $\epsilon_{\text{mc}}^{K_s^0 ee}$, $\epsilon_{\text{mc}}^{K^+ ee}$, f'_{tracking}, f_{tracking}, f_{kaonID}, $f_{K_s^0}$.
We assumed there are no correlation between these quantities, and hence the correlation term is missing in the equation. But if some generic systematic used like for tracking then correlation term will make it a full square. Like for example here, $3\sigma_{\text{tracking}}$ used for tracking where σ_{tracking} is common systematic uncertainty for per track.

A set of variable to another set of variables...

Suppose there are m similar functions, $y_1(\vec{x})$, $y_2(\vec{x})..y_m(\vec{x})$,

$$U_{kl} = \text{cov}[y_k, y_l] = \sum_{i,j=1}^{n} \left[\frac{\partial y_k}{\partial x_i} \frac{\partial y_l}{\partial x_j} \right]_{\vec{x}=\vec{\mu}} V_{ij} \tag{4}$$

In short,

$$U = AVA^T \tag{5}$$

where A is Jacobian from $\vec{x} \to \vec{y}$ or $A_{ij} = \left[\frac{\partial y_i}{\partial x_j} \right]_{\vec{x}=\vec{\mu}}$

$$A_{m\times n} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix}$$

Propagation of a covariance matrix (V) of a set of variables $\vec{x}$ to another covariance matrix (U) of another set of variables $\vec{y}$, where y's are functionally connected to x's

$$U_{m\times m} = A_{m\times n} V_{n\times n} A^T_{n\times m} \tag{6}$$

Q. A tracking chamber finds the hit coordinates in cylindrical polar coordinates (ρ, ϕ, z). The uncertainty in the ρ direction is negligible. Find the covariance matrix for the position in Cartesian coordinates(x, y, z)

A.

1 $\vec{x} = (\rho, \phi, z)$ and $\vec{y} = (x, y, z)$

2 $x = \rho\cos\phi$, $y = \rho\sin\phi$ and $z = z$

3 $V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_\phi^2 & 0 \\ 0 & 0 & \sigma_z^2 \end{pmatrix}$

- $A = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- $U = AVA^T$

Q. A common resistor, with resistance, R and a voltmeter is used to measure the potential difference across the resistor, V_1 and V_2, from two power supplies applied in turn. The uncertainty on the resistance is S_R. The uncertainty in the voltage supplies has two components, one is statistical uncorrelated, which are σ_{V_1}, σ_{V_2}, another is fully correlated S_V. Construct covariance matrix for $(R,\ V_1,\ V_2\ )$ and then find for (I_1,I_2)

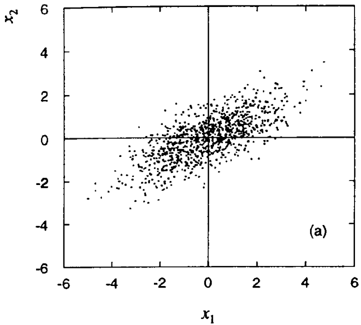
A.

1. $V = \begin{pmatrix} S_R^2 & 0 & 0 \\ 0 & S_V^2 + \sigma_{V_1}^2 & S_V^2 \\ 0 & S_V^2 & S_V^2 + \sigma_{V_2}^2 \end{pmatrix}$
2. $A = \begin{pmatrix} \frac{\partial I_1}{\partial R} & \frac{\partial I_1}{\partial V_1} & \frac{\partial I_1}{\partial V_2}(=0) \\ \frac{\partial I_2}{\partial R} & \frac{\partial I_2}{\partial V_1}(=0) & \frac{\partial I_2}{\partial V_2} \end{pmatrix}$

Combination of measurements for uncorrelated source

$$x = \frac{\sum_{i=1} \frac{x_i}{\sigma_i^2}}{\sum_{i=1} \frac{1}{\sigma_i^2}}, \text{ find } \sigma_x$$

PCA



- Reduce the dimension of the above distribution

$\rightarrow$ Project, along an axis where the distribution is maximally scattered

$\rightarrow$ Next few slides will achieve that target

Diagonalization of covariance matrix

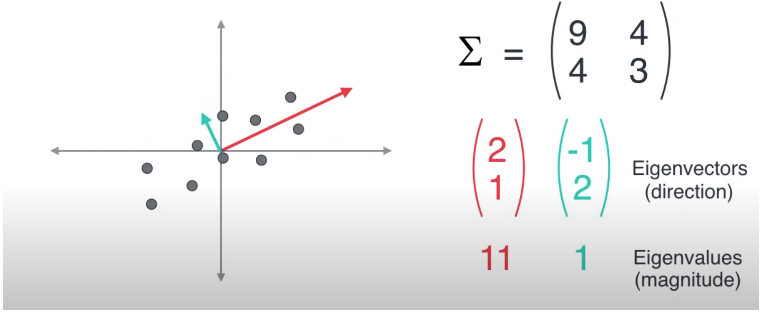
$$U = AVA^T \tag{7}$$

- For particular, choice of A, U can be diagonal.
- Once we diagonalize, A can be written with the help of normalized eigenvectors $A = \begin{pmatrix} \cdots & \alpha^1 & \cdots \\ \cdots & \alpha^2 & \cdots \\ \cdots & \alpha^3 & \cdots \\ & \vdots & \\ \cdots & \vdots & \cdots \end{pmatrix}$

$$y_k(\vec{x}) - y_k(\vec{\mu}) \approx \sum_{i=1}^{n} \left[\frac{\partial y_k}{\partial x_i}\right]_{x=\vec{\mu}} (x_i - \mu_i) \tag{8}$$

- So $y_k(\vec{x}) - y_k(\vec{\mu}) \approx A_{ki}(x_i - \mu_i)$ or $\begin{pmatrix} y_1 - \bar{y}_1 \\ y_2 - \bar{y}_2 \\ y_3 - \bar{y}_3 \\ \vdots \\ y_n - \bar{y}_n \end{pmatrix} = \mathrm{A} \begin{pmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \\ x_3 - \bar{x}_3 \\ \vdots \\ x_n - \bar{x}_n \end{pmatrix}$
- So we get n a new set of variables amongst which the correlation is zero.
- y_k is related to α^k (Sort w.r.t eigenvalue magnitude) eigenvector

Contd..



- y_k is related to α^k (Sort w.r.t eigenvalue magnitude in descending order) eigenvector
- Prune y's from bottom to reduce dimension

Ref. Principal Component Analysis (PCA)


THANK
YOU

Backup

Detailed Calculation

$$X = \frac{2a \times b \times c - d}{2a \times b \times c + d}$$
$$= \frac{N}{D}, \text{ with} N = 2abc - d \text{ and } D = 2abc + d$$

Now,

- $\frac{\partial X}{\partial a} = \frac{1}{D}\frac{\partial N}{\partial a} + \frac{N}{-D^2}\frac{\partial D}{\partial a} = \frac{2bc}{D} - \frac{N}{D^2}2bc = \frac{2bc}{D}(1 - \frac{N}{D}) = \frac{2bc}{D}\frac{D-N}{D} = \frac{4bcd}{D^2} = \frac{1}{a}\frac{4abcd}{D^2}$
- In a similar way, $b\frac{\partial X}{\partial b} = \frac{4abcd}{D^2} = c\frac{\partial X}{\partial c}$
- $\frac{\partial X}{\partial d} = \frac{1}{D}\frac{\partial N}{\partial d} + \frac{N}{-D^2}\frac{\partial D}{\partial d} = \frac{1}{D} \times (-1) - \frac{N}{D^2} = \frac{1}{D}(1 + \frac{N}{D}) = \frac{1}{D}\frac{D+N}{D} = \frac{4abc}{D^2} = \frac{1}{d}\frac{4abcd}{D^2}$

Continued ..

$$\sigma_X = \sqrt{\left(\frac{\partial X}{\partial a}\sigma_a\right)^2 + \left(\frac{\partial X}{\partial b}\sigma_b\right)^2 + \left(\frac{\partial X}{\partial c}\sigma_c\right)^2 + \left(\frac{\partial X}{\partial d}\sigma_d\right)^2 + \left(\sum_{i,j=\{a,b,c,d\}} \frac{\partial X}{\partial x_i}\frac{\partial X}{\partial x_j}V_{ij}\right)}$$

$$= \frac{4abcd}{D^2}\sqrt{\left(\frac{\sigma_a}{a}\right)^2 + \left(\frac{\sigma_b}{b}\right)^2 + \left(\frac{\sigma_c}{c}\right)^2 + \left(\frac{\sigma_d}{d}\right)^2 + \left(\sum_{i,j=\{a,b,c,d\}} \frac{V_{ij}}{x_i x_j}\right)}$$

If efficiency combined..

- $\epsilon^{\text{mc}}_{K\ell\ell} = f\epsilon^{\text{mc}}_{K\mu\mu} + (1-f)\epsilon^{\text{mc}}_{Kee}$ and $\epsilon^{\text{dt}}_{K\ell\ell} = f\epsilon^{\text{dt}}_{K\mu\mu} + (1-f)\epsilon^{\text{dt}}_{Kee}$,
- $\epsilon^{\text{dt}}_{K\ell\ell}/\epsilon^{\text{mc}}_{K\ell\ell} = r_{\text{track}} r_{\text{mva}} r_{\text{kID}} \frac{f\epsilon^{\text{dt}}_{0,Kee}\epsilon^{\text{dt}}_{\text{electronID}} + (1-f)\epsilon^{\text{dt}}_{0,K\mu\mu}\epsilon^{\text{dt}}_{\text{muonID}}}{f\epsilon^{\text{mc}}_{0,Kee}\epsilon^{\text{mc}}_{\text{electronID}} + (1-f)\epsilon^{\text{mc}}_{0,K\mu\mu}\epsilon^{\text{mc}}_{\text{muonID}}}$ $\longrightarrow$very complicated