School of Mathematics
Tata Institute of Fundamental Research

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NOTICE

Speaker	:	***Andrei Lavrenov***
Affiliation	:	***St. Petersburg State University, Russia***
Title	:	***Different definitions of unstable orthogonal*** K_2
Date & Time	:	***Friday, 17 June, 2022 at 2.00 p.m.***
Venue	:	***Lecture Room (AG-77)***

Abstract

Many approaches to higher algebraic K-theory and Hermitian K-theory are known. For example, stable Quillen's groups $\mathrm{K}_n(R)$ (defined e. g. via the $+$-construction) and stable Karoubi–Villamayor groups $\mathrm{KV}_n(R)$ (defined via standard simplicial scheme) coincide for $n \geq 1$ if R happens to be a regular ring. These theories use infinite-dimensional algebraic groups such as $\mathrm{GL}_\infty(R)$ in their definition. In this talk we will discuss an *unstable* analogue of such result for the functor K_2.

The interest for the unstable Quillen's K_2-groups, in particular, comes from the fact that they appear in Steinberg's presentation of the groups of points of algebraic groups by means of generators and relations. On the other hand, Karoubi–Villamayor K_2-groups can be interpreted as fundamental groups in the unstable $\mathbb{A}^1$-homotopy category $\mathscr{H}_\bullet(k)$ of F. Morel and V. Voevodsky (using results of A. Asok, M. Hoyois and M. Wendt). Conjecturally, for any split simple group $G = \mathrm{G}(\Phi, R)$ with $\mathrm{rk}\,\Phi \geq 5$ and regular ring R holds an equality

$$\pi_1^{\mathbb{A}^1}(G)(R) = \pi_2(\mathrm{B}G^+). \tag{1}$$

We remark that the Nisnevich localization $\mathrm{a}_{\mathrm{Nis}}\,\pi_1^{\mathbb{A}^1}(G)(R)$ of $\mathbb{A}^1$-fundamental groups was recently computed by F. Morel and A. Sawant, and coincides with the unramified Milnor $\underline{\mathrm{K}}_2^{\mathrm{M}}$ or Milnor-Witt $\underline{\mathrm{K}}_2^{\mathrm{MW}}$ sheaf depending on Φ.

Conjecture (1) is parallel to the Serre's problem and Bass–Quillen conjecture, and we adopt Quillen–Suslin and Lindel–Popesque results for this case. In particular, for $\Phi = \mathsf{A}_l$, D_l this conjecture is already proven for a regular ring R containing a field k of characteristic $\neq 2$, $l \geq 7$.

As a corollary, one can obtain the following results.

- The group $\mathrm{Spin}_{2l}(k[t_1,\ldots,t_n])$ admits an explicit presentation by means of generators and relation (generalizing Steinberg's presentation in the case $n = 0$).
- $\mathrm{H}_2\big(\mathrm{Spin}_{2l}(k[t_1,\ldots,t_n]), \mathbb{Z}\big) = \mathrm{K}_2^{\mathrm{M}}(k)$.
- $\mathrm{H}_2(\mathrm{O}_{2l}(R[t]), \mathbb{Z}) = \mathrm{H}_2(\mathrm{O}_{2l}(R), \mathbb{Z})$.

The talk is based on my joint work with Sergey Sinchuk and Egor Voronetsky.

Milind Pilankar